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Design of Seasonal Adjustment Filter Robust to Variations in the Seasonal Behaviour of Time Series

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Considering that many macroeconomic time series present changing seasonal behaviour, there is a need for filters that are robust to such changes. This article proposes a method to design seasonal filters that address this problem. The design was made in the frequency domain to estimate seasonal fluctuations that are spread around specific bands of frequencies. We assessed the generated filters by applying them to artificial data with known seasonal behaviour based on the ones of the real macroeconomic series, and we compared their performance with the one of X-13A-S. The results have shown that the designed filters have superior performance for series with pronounced moving seasonality, being a good alternative in these cases.

Key words: Moving seasonality; Filter design; Frequency domain; Time series decomposition; X-13A-S.

1. Introduction

Changing seasonality of time series was first noted in the nineteenth century (Gilbart 1852, quoted in Bell and Hillmer 1984), and is common in macroeconomic data (Canova and Ghysels 1994; Wells 1997; Franses and Koehler 1998; Van Dijk et al. 2003). Such changes can be due to variations in seasonal amplitude from year to year or in the proportionality relationship between the seasonal at each month and the seasonal at each other month (i.e., the seasonal pattern) (Godfrey and Karreman 1964). We refer to them as ‘moving seasonality’.

Kuznets (1932) was among the first authors to highlight the importance of moving seasonality. Since then, statistical tests have been created to evaluate the presence of changing seasonal behaviour (Higginson 1975; Canova and Hansen 1995; Sutradhar and Dagum 1998) and several seasonal adjustment methods have been suggested to tackle it, many of them developed in the frequency domain. Among the frequency domain approaches, we highlight the pioneering work of Hannan (1964), Nerlove (1964), and Nettheim (1964).

X-13ARIMA-SEATS (X-13A-S) is the most recent enhanced version of the ‘X-11 family’ (U.S. Census Bureau 2013). This program contains two seasonal adjustment

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modules: the X-11 method and the SEATS. The latter is a seasonal adjustment procedure that follows the ARIMA model-based signal extraction technique (Gómez and Maravall 1996). The former module is the X-11, or Census X-11, one of the most commonly used methods for seasonal adjustment of economic time series used by government agencies and statistical bureaus. This method, based on moving averages, was introduced in 1965 by the U.S. Census Bureau (Shiskin et al. 1967) and further contributions have been added to the basic version (Dagum 1980; Dagum 1988; Findley et al. 1998). It is important to mention that these methods are also implemented in JDEMETRA+, which is the software officially recommended by Eurostat and the European Central Bank for the seasonal and calendar adjustment of official statistics in the European Union. The ESS guidelines on seasonal adjustment (Eurostat 2015) highlight the unstable seasonality problem, warning that the standard seasonal adjustment cannot be used in this case.

In the literature, there are several works comparing X-11 with other methods of seasonal adjustment, especially with SEATS (Hood et al. 2000; Findley 2005; Tiller et al. 2007). The results point to similar performance when the time series presents common seasonal behaviour. However, in cases of data with moving seasonality, the X-11 method has some drawbacks (Planas 1998; Kaiser and Maravall 2000; Maravall and Pérez 2011).

Nettheim (1965) listed strategies for dealing with moving seasonality. One of them is a filter designed to have unit gain around each seasonal frequency and very small gain elsewhere. As pointed out by the author, a drawback of such a method is that one should determine in advance how wide the unit gain region should be. A way of circumventing this problem is to use spectral estimation methods. Examples are the non-ad-hoc methods in Melnick and Moussourakis (1974) and Geweke (1978). The ARIMA model-based approach of SEATS (Gómez and Maravall 1996) is another attempt to treat moving seasonality, but in some cases it is not trivial to find a good-fitting model with a valid decomposition into components (Tiller et al. 2007). The Structural Models (STM) (Koopman et al. 2000) can deal with moving seasonality via a sophisticated model-based approach that requires expert operators. However, the simplicity of the seasonal adjustment programs is sometimes preferred to seasonally adjusting a large number of series.

In this context, considering that the seasonal adjustment is largely used in the production of official statistics, we propose a methodology to design seasonal filters to deal with moving seasonality. The design of such filters, which we refer to as Seasonal-WLS (S-WLS), is based on least squares criteria in the frequency domain. The design is inspired by the requirements set forth in Nettheim (1965). We assess the performance of the proposed S-WLS filters by running them on artificial data derived from the behaviour of the real macroeconomic time series. Then, we compared its adjustment with the adjustment of the X-11 method. We make this comparison because, besides the fact that X-11 tends to misadjust series with highly moving seasonality, it is ad-hoc and has been one of the most widely used seasonal adjustment methods.

This article is organised as follows. Section 2 briefly describes the theoretical framework of the X-11 method, as well as the frequency domain representation of its filters. Section 3 presents the proposed method to design the S-WLS filters for seasonal adjustment, describing its structure and the parameters' choice. Section 4 shows the results of the application of the proposed S-WLS filters, comparing their performance with that of X-11. Finally, Section 5 summarises and discusses the main findings. Appendix A presents the X-11 algorithm in the frequency domain, and Appendix B and C present, respectively, the details about the selection of the filter parameters and about the signal-to-noise ratio (SNR) computation.

2. The X-11 Method in the Frequency Domain

The X-13ARIMA-SEATS (X-13A-S) program contains the implemented X-11 seasonal adjustment method. This method, as well as other programs of the 'X-11 family', consists of a moving average procedure for seasonally adjusting series (it is fully explained in Findley et al. 1998). The frequency domain properties of the X-11 method have been discussed in the works of Wallis (1982), Bell and Monsell (1992), Dagum et al. (1996), Gómez and Maravall (2001), Findley and Martin (2006) and others. Here, the X-11 procedure will be briefly discussed for the purpose of introducing its transfer function, which will be instrumental in analysing the X-11 behaviour in the presence of moving seasonality.

The seasonal adjustment filters of X-11 are available in X-13A-S (U.S. Census Bureau 2013), X-12-ARIMA (Findley et al. 1998) and X-11-ARIMA (Dagum 1980). In the literature, the hybrid name 'X-11/12-ARIMA filters' was adopted to designate these filters (Findley and Martin 2006). In this work, they will be referred to just as 'X-11 filters'.

The step-by-step application of the X-11 method (default setting) can be summarised in two stages, for seasonal factor and seasonal adjustment (Findley et al. 1998). In the default procedure, it specifies a 3 × 3 seasonal moving average (usually called 'seasonal filter'), $M_{3\times3}$, for the initial seasonal factor estimates, and the 3 × 5 seasonal filter ($M_{3\times5}$) thereafter. It also prefilters the input series with a 2 × 12 moving average, ($M_{2\times12}$). The whole procedure is depicted in Figure 1, considering the additive decomposition of monthly series, where: Y is the original time series; T is the trend estimate; SI is the estimate of the seasonal-irregular; $\hat{S}$ is the preliminary seasonal factor; S is the seasonal factor; A is the seasonally adjusted time series. The superscript$^{(1)}$ means the initial estimate and the superscript$^{(2)}$ refers to the final one. H_{13} is the 13-term Henderson trend filter.

The coefficients of the seasonal filters present in X-11, as well as the 2 × 12 moving average, are listed in Tables 1 and 2.

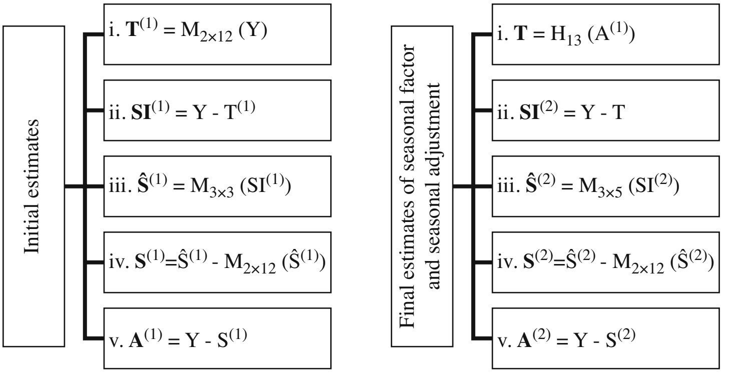

Fig. 1. X-11 default procedure for seasonal adjustment considering the additive decomposition of monthly series.

Table 1. Coefficients of the X-11 Seasonal Filters ($m(n) = m(-n)$): $m(n)$ are the coefficients of filter $M_{PXQ}(z)$ (see Equation (A.1) in Appendix A).

Seasonal filters	m(5)	m(4)	m(3)	m(2)	m(1)	m(0)
3×9	1/27	2/27	3/27	3/27	3/27	3/27
3×5			1/15	2/15	3/15	3/15
3×3				1/9	2/9	3/9

In the automatic selection procedure, the program may replace the 3×5 seasonal moving average filter in step (iii) of the 'final estimates of seasonal factor' in Figure 1, by either a 3×3 or a 3×9 seasonal filter (Findley et al. 1998; U.S. Census Bureau 2013). Regarding the Henderson trend filter, the program selects a trend moving average based on statistical characteristics of the data. For monthly data, either a 9-, 13-, or 23-term Henderson trend filter can be selected by the automatic procedure.

The procedure used to compute the seasonally adjusted series ($A^{(2)}$) described in Figure 1 can be expressed in the frequency domain by the following expression:

$$A^{(2)} = Y(z)\{1 - \mathrm{M}_{3\times5}(z^{12})[1 - \mathrm{M}_{2\times12}(z)][1 - H_{13}(z) \\ \times \{1 - [1 - \mathrm{M}_{2\times12}(z)]^2 \mathrm{M}_{3\times3}(z^{12})\}]\} \tag{1}$$

where the functions of z are the z-transforms of the corresponding filters. The coefficients $m(n)$ for each filter are listed in Tables 1 and 2. Detailed explanation about this expression is given in Appendix A.

The expression in Equation (A.12) provides a useful way to evaluate the transfer function of the various X-11 filters, both for the default and the optional choices in the automatic procedure. Figure 2 shows the magnitude of the transfer functions of the X-11 for the three types of seasonal moving average, considering a 13-term Henderson filter and monthly data.

Figure 2 illustrates the fact that the smaller the size of the seasonal filter, the wider its passband width is, and the more suitable it is for treating moving seasonality data (for more details, see Subsection 3.2). However, even the smallest seasonal filter in the automatic option of the seasonal adjustment (3×3) does not have a large enough passband in order to deal with moving seasonality. The X-11 method also provides the possibility of using a three-term moving average filter, although it is not available in the automatic procedure; in addition, it produces a transfer function with a poor attenuation in the stopband.

Table 2. Coefficients of the X-11 Moving Average Filter ($m(n) = m(-n)$): $m(n)$ are the coefficients of filter $M_{PXQ}(z)$ (see Equation (A.1) in Appendix A).

Moving average	m(6)	m(5)	m(4)	m(3)	m(2)	m(1)	m(0)
2×12	1/24	1/12	1/12	1/12	1/12	1/12	1/12

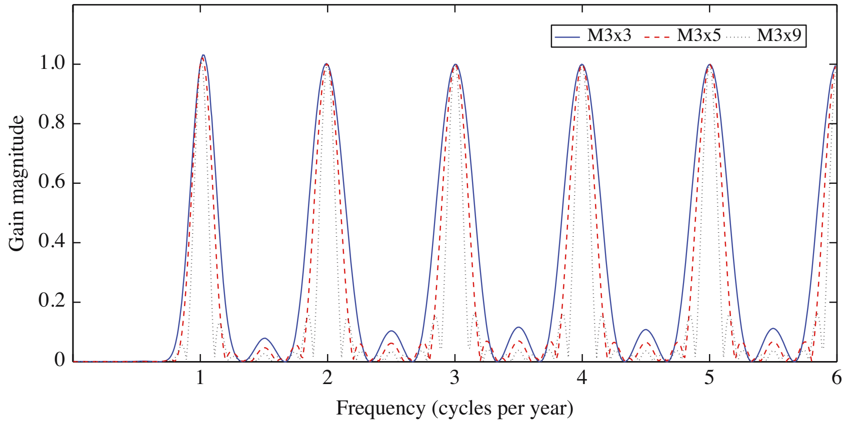

Fig. 2. Magnitude of the transfer function of the X-11 for different seasonal filters: monthly series.

In this article we propose a design method that generate filters with bandwidths that are large enough so that they can deal with the most common seasonality variations, without compromising the filter attenuation outside the seasonal frequencies. The construction of this filter will be presented in the next section.

3. The Proposed Seasonal Filter: a Frequency Domain Moving Seasonal Filter to Deal With Moving Seasonality

To introduce the proposed seasonal filters, we assume that a monthly observable time series at time t, $Y(t)$, can be represented as follows:

$$Y(t) = T(t) + S(t) + I(t), (t = 1, 2, \ldots) \quad (2)$$

where $Y(t)$ is the original time series, $T(t)$, $S(t)$, and $I(t)$ are unobservable trend-cycle (treated here as 'trend'), seasonal and irregular components.

From a frequency domain point of view, a filter designed to extract the seasonality should be able to isolate the movements in the series which occur in the seasonal frequency and in its harmonics, usually called 'seasonal frequencies': ($2\pi/12$, $4\pi/12, \ldots, 12\pi/12$). However, when the series has moving seasonality, its spectral mass is not restricted to the seasonal frequencies, but is spread around their neighborhoods. Considering this, we want a filter with frequency response equal to one in the bands around the seasonal frequencies (passbands) and zero in the remaining frequencies (stopbands). This is one of the filters mentioned by (Nettheim 1965), illustrated in Figure 3.

This filter has the objective of not disturbing the frequency components around the harmonics of the seasonal frequency, and to this end its transfer function has a flat shape in a neighbourhood of width Δ around these frequencies, as shown in Figure 3. This is important in the case of moving seasonality. An example that illustrates this situation is given by the following time series, composed by an irregular component and a seasonal

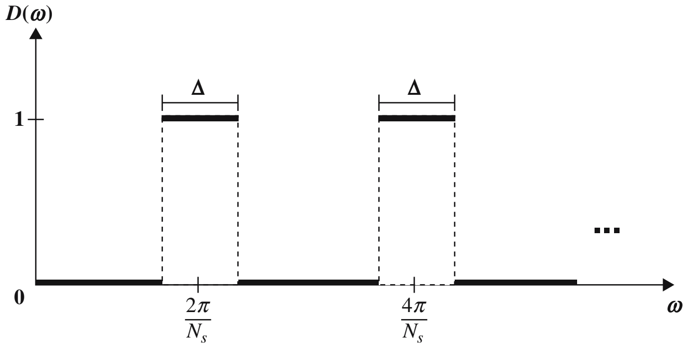

Fig. 3. Magnitude of the transfer function of the ideal filter.

component with nonstationary changes:

$$Y(t) = \sum_{i=1}^{P} \left[1 + b \sin\left(2\pi \frac{(t - iQ)}{k_i} \right) \right] \left[\sin 2\pi \frac{t}{2} \right] + I(t) \tag{3}$$

where $b = 0.9$, $Q = 120$, k_i are samples from a random variable uniformly distributed in the interval [70, 240] and $I(t)$ is the irregular component, an independent zero-mean Gaussian process.

The spectrum of this time series is shown in Figure 4. It is possible to note that the seasonal component has significant energy over a bandwidth of approximately 0.03 around the frequency 1/12 cycles/month. In order not to attenuate the frequency components that deviate from the harmonic of the seasonal frequencies, the gain of the filter should be constant for all the frequencies in the neighbourhood of the seasonal frequencies. This block shape has been suggested by Nettheim (1965).

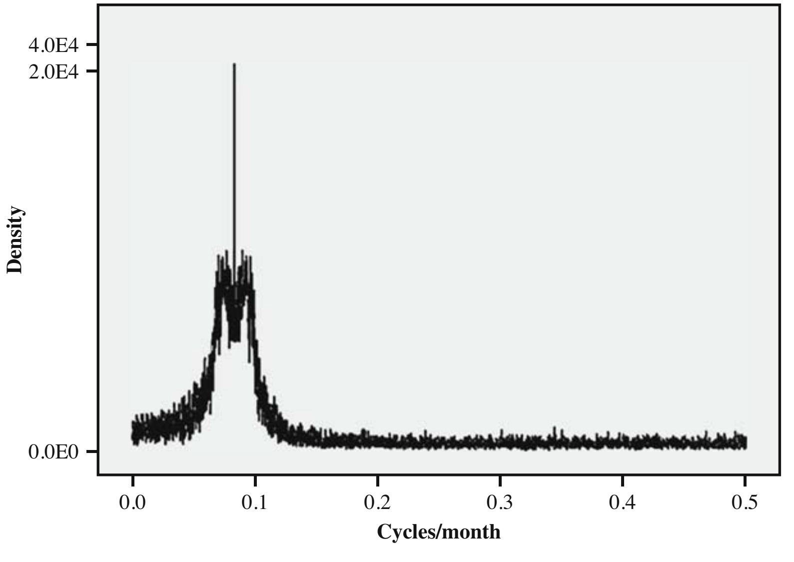

Fig. 4. Spectral density of Y(t) by frequency.

Such a filter can be designed to accommodate the different kinds of seasonality variations. These can be seen as combinations of variations in the seasonal periods as well as variations in the seasonal amplitude. We should take these into account when computing the filter parameters. To this end, one can express a seasonal component with frequency ω_s and moving seasonality as

$$s(t) = [1 + a(t)]h(t), \tag{4}$$

where h is a periodic function with period $(\omega_s + \Delta\omega)$ and $\Delta\omega$ may vary with time t; $a(t)$ represents the seasonal amplitude variation.

Considering that the rate of variation of $\Delta\omega$ is much smaller than the seasonal period, $h(t)$ can be expressed by a Fourier series as:

$$h(t) = \sum_n c_n e^{jn[\omega_s + \Delta\omega(t)]t} \tag{5}$$

From Equation (4), the seasonal signal in the frequency domain can be written as:

$$S(\omega) = \sum_n 2\pi c_n \delta(\omega - n\omega_s - n\Delta\omega) + \sum_n 2\pi c_n A(\omega - n\omega_s + n\Delta\omega) \tag{6}$$

where $A(\omega)$ is the Fourier transform of $a(t)$ and $\delta(\omega)$ is the Dirac delta function.

Equation (6) is depicted in Figure 5, where the arrows indicate the Dirac delta functions and the bell-shaped functions are repetitions of $A(\omega)$ centred at the frequencies $n\omega_s + n\Delta\omega$. From this, since the bandwidth of $a(t)$ is given by B, and considering that one has to account for up to the nth harmonic component, the width Δ from each of the filter's passbands has to be larger than:

$$\Delta = 2\max\{n\Delta\omega + B, -n\Delta\omega + B\} = 2(n|\Delta\omega| + B) \tag{7}$$

Therefore, to perform the seasonal adjustment it is necessary to determine the filter design parameter Δ, that is, a function of the seasonal behaviour of the series. In practice, this can be done, for example, by computing the series spectrogram in order to determine ω_s, $\Delta\omega$, and B.

Regarding what was discussed in this subsection, we propose a methodology to design seasonal filters that are able to deal with moving seasonality, that is, with transfer functions

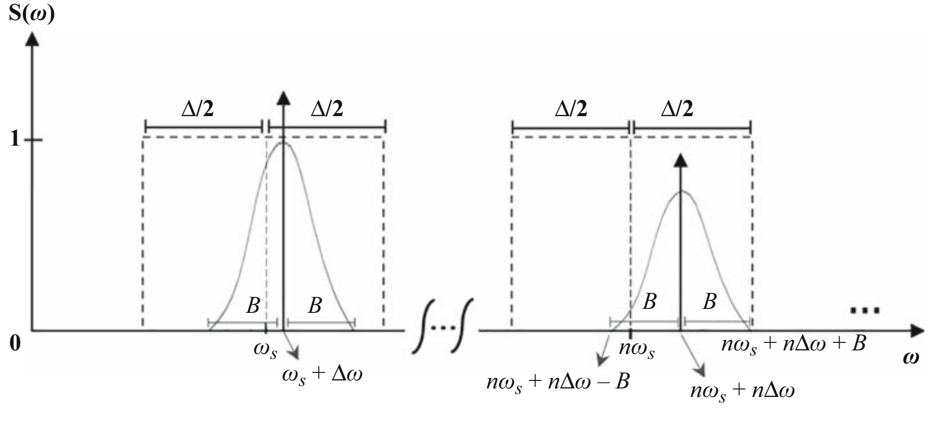

Fig. 5. Magnitude of the ideal filter transfer function and spectrum of the seasonal signal from Equation (6).

that approximates the one in Figure 3. We refer to them as S-WLS (Seasonal Weighted Least Squares) filters. These filters are finite and symmetric, designed in the frequency domain, and can be applied to a monthly seasonal time series, independent of its distribution.

Applying the proposed S-WLS filters, the data will be seasonally adjusted by eliminating the trend component and performing the seasonal extraction using a single filtering operation. The following subsection presents the design of the S-WLS filters. The theory used in this section is based on Diniz et al. (2010).

3.1. The Structure of the S-WLS Filters

First of all, to extract the seasonality, it is necessary to also eliminate the trend component (Hassani 2007; Cleveland et al. 1990; Burman 1980).

With this purpose in mind, the z-transform of the filter frequency response should have a term of the form $(1 - z^{-1})^{j+1}$, that accounts for eliminating a trend polynomial up to order j.

Therefore, we can represent the S-WLS filters, to extract the seasonal component, by the following z-transform:

$$P(z) = (1 - z^{-1})^{j+1} G(z) \tag{8}$$

where $G(z)$ is defined as

$$G(z) = \sum_{t=-p}^{L-p-1} g(t) z^{-t}. \tag{9}$$

In Equation (9), L is the number of degrees of freedom of the filter, given by the coefficients $g(t) \in \mathbb{R}$; the filter length is $(L + j + 1)$; p gives a shift in the filter output, and for a filter with zero delay, it should be equal to $(L + j + 1)/2$. The index t represents the time period $(t = 1, 2, \ldots)$.

The coefficients $g(t)$ of $G(z)$ must be optimised so that the frequency response of the filter can approximate the desired frequency response $D(\omega)$. In other words, $G(z)$ is adjusted so that the resulting filter can approximate the one from Nettheim (1965) (illustrated in Figure 3) with bandwidths around the harmonics of the seasonal frequency as flat as possible. Besides, to make the filter robust to variations in seasonality, these coefficients must be optimised to consider the seasonal variation around the harmonics in a frequency range corresponding to a percentage of the seasonal frequency. Moreover, it must suppress as much of the irregular component as possible. Thus, in the S-WLS design, the passbands have a desired frequency response $(D(\omega))$ equal to one and the stopbands have a desired frequency response equal to zero.

In addition, in order to help in the optimisation process, we introduce 'don't care' bands, where the desired response is not specified, between each adjacent passband and stopband. Their width is adjusted experimentally so that the obtained frequency response is as close as possible to the desired one. These design parameters are illustrated in Figure 6, where N_s indicates the assumed seasonal period, which is twelve for monthly data.

As can be observed in Figure 4, the filter is robust to seasonality variations up to a fraction $(\alpha/2)$ of the assumed seasonal frequency.

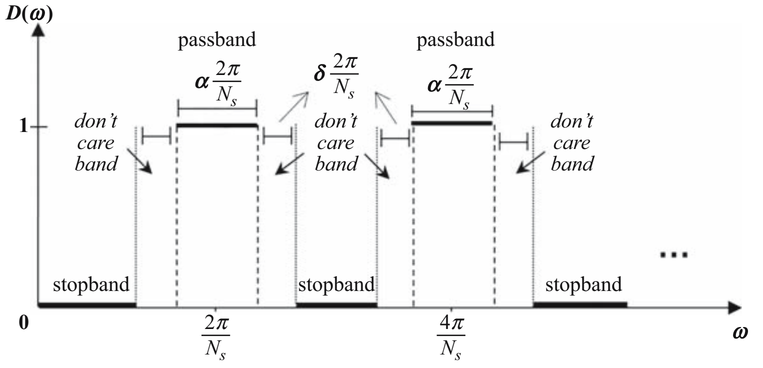

Fig. 6. Magnitude of the transfer function of the proposed S-WLS filter.

The filter coefficients are obtained by an optimisation process, which minimises the Euclidean distance between the desired frequency response $D(\omega)$ and the filter frequency response $P(e^{i\omega})$.

Since the frequency response of the filter can be computed from its z-transform by making $z = e^{i\omega}$ (Diniz et al. 2010), we have that, from Equation (8), it becomes

$$P(e^{i\omega}) = (1 - e^{-i\omega})^{j+1} G(e^{i\omega}) \tag{10}$$

$$= e^{-i\left(\frac{\omega}{2}\right)(j+1)} \left(2i \sin\frac{\omega}{2}\right)^{j+1} G(e^{i\omega})\,. \tag{11}$$

Since $G(z)$ can be written as

$$G(e^{i\omega}) = e^{-i\omega(p)} \mathbf{E}^{\mathrm{T}}(\omega)\mathbf{g} \tag{12}$$

where $\mathbf{E}(\omega) = \left[1 \;\; e^{-i\omega} \;\; \ldots \;\; e^{-i\omega(L-1)}\right]^{\mathrm{T}}$ and $\mathbf{g} = \left[g(-p) \;\; g(-p+1) \;\; \ldots \;\; g(L-p-1)\right]^{\mathrm{T}}$, the filter frequency response becomes

$$P(e^{i\omega}) = e^{-i\left(\frac{\omega}{2}\right)(j+1) - i\omega(p)} \left(2i \sin\frac{\omega}{2}\right)^{j+1} \mathbf{E}^{\mathrm{T}}(\omega)\mathbf{g} \tag{13}$$

$$= s(\omega, j, p)\mathbf{E}^{\mathrm{T}}(\omega)\mathbf{g}. \tag{14}$$

where $s(\omega, j, p) = e^{-i\omega\left[\left(\frac{j+1}{2}\right)+p\right]} \left(2i \sin\frac{\omega}{2}\right)^{j+1}$.

In the optimisation process, we will discretise the frequency variable ω in the passband and in the stopband. Thus, it is relevant to consider the possibility that the errors in the passbands and in the stopbands have different importance. To allow this in the optimisation process, we assign a weight $W(\omega)$ to each frequency. It establishes the relative importance of the frequency response at each frequency ω during the optimisation. For example, if we assign a higher importance to the error in the passband, the transfer function would tend to be like the one in Figure 7a. In contrast, if the importance of attenuation in the stopband is much higher than the one in the passband, we would tend to have the transfer function like the one in Figure 7b. As can be observed, the transfer function may change considerably.

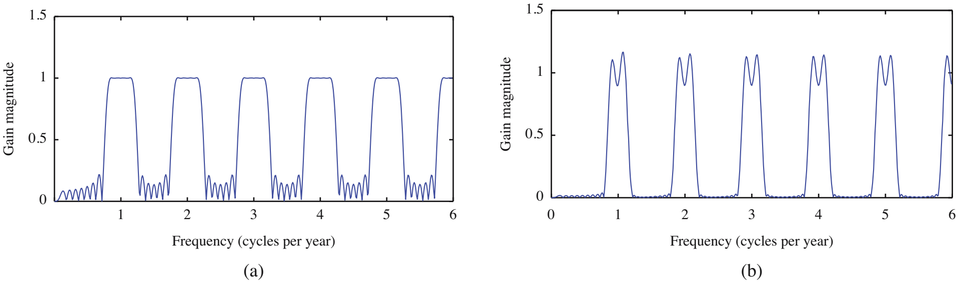

Fig. 7. Magnitude of the transfer function of the S-WLS filter when $N = 145$, $\alpha = 1/3$, $\delta = 1/30$ and (a) $w_0 = 30$. (b) $w_0 = 0.05$.

Formally, such frequency response weighting is equivalent to minimising the average of the weighted squared error below:

$$|e_r(\omega)|^2 = |[P(\omega) - D(\omega)]W(\omega)|^2 \tag{15}$$

where $D(\omega)$ is the desired frequency response (see Figure 6).

In order to perform this minimisation we discretise ω at the set of frequencies $(\omega_1, \omega_2, . . . , \omega_n)$. The number n of frequency samples is equal to $401N$, where N is the filter order. Therefore, each of the functions of ω can be represented as a column vector consisting of the samples of the function at the discrete set of frequencies. For example, we represent $P(\omega)$ as

$$\mathbf{P} = \begin{bmatrix} P(\omega_1) & P(\omega_2). . . & P(\omega_n) \end{bmatrix}^{\mathrm{T}}. \tag{16}$$

Using this notation, Equation (14) is equivalent to:

$$\mathbf{P} = \mathbf{Ug} \tag{17}$$

where $\mathbf{P}$ is defined in Equation (16) and the matrix $\mathbf{U}$ of dimensions $n \times L$ is defined as

$$\mathbf{U} = \begin{bmatrix} \mathbf{E}^{\mathrm{T}}(\omega_1)s(\omega_1, j, p) \\ \mathbf{E}^{\mathrm{T}}(\omega_2)s(\omega_2, j, p) \\ . . . \\ \mathbf{E}^{\mathrm{T}}(\omega_n)s(\omega_n, j, p) \end{bmatrix}. \tag{18}$$

If the samples of error (ω) and the desired frequency response $D(\omega)$ are represented analogously as column vectors, and we define

$$\mathbf{W} = \begin{bmatrix} \mathrm{w}_1(\omega_1) & 0 & 0 & 0 \\ 0 & \mathrm{w}_2(\omega_2) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & . . . & \mathrm{w}_n(\omega_n) \end{bmatrix}, \tag{19}$$

then, from Equation (17) to (19), Equation (15) can be expressed in matrix form as

$$\mathbf{e_r} = \mathbf{W}[\mathbf{P} - \mathbf{D}] = \mathbf{W}[\mathbf{Ug} - \mathbf{D}]. \tag{20}$$

The sum of squared errors can be written as

$$\|\mathbf{e_r}\|_2^2 = \|\mathbf{e_r^T e_r}\| = (\mathbf{Ug} - \mathbf{D})^{*T} \mathrm{W}^{*T} \mathrm{W} (\mathbf{Ug} - \mathbf{D}) \tag{21}$$

which is minimized by the vector

$$\mathbf{g} = \left(\mathbf{U}^{*T}\mathbf{W}_s^2\mathbf{U} + \mathbf{U}^T\mathbf{W}_s^2\mathbf{U}^*\right)^{-1}(\mathbf{U}^T + \mathbf{U}^{*T})\mathbf{W}_s^2\mathbf{D}. \tag{22}$$

Convolving the vector $\mathbf{g}$ with the coefficients of the polynomial $(1 - z^{-1})^{j+1}$ (Equation (8)), we obtain the vector with the S-WLS filter coefficients. The filtering operation is accomplished by convolving the vector of the S-WLS coefficients with the time series. The output of this operation is the extracted seasonal component. The adjusted series is obtained by subtracting this result from the original series.

The S-WLS filters have five design parameters (see Figure 6):

(i) the parameter α is equivalent to the bandwidth around the seasonal frequencies, being related to the seasonal stability (it depends on the data characteristics);
(ii) the parameter δ is related to the width of the 'don't care' band, helping in the optimisation process;
(iii) the weight (w_0) indicates the importance given to the error minimisation in the passbands compared with the one in the stopbands – large values of weight (w_0) result in gain close to 1 around the seasonal frequencies, but the attenuation outside the seasonal frequencies decreases;
(iv) the filter size N, representing the number of coefficients of the filter;
(v) the number of frequency samples used during the optimisation. In the filter experiments we used $401N$ because it was shown to be enough to provide a good approximation.

Considering a fixed value for the parameter N, and to a given α, different δ and w_0 lead to considerable changes in the filter transfer function. Figures 8a to 8d show the transfer function for some values of the parameters δ and w_0 for $\alpha = 1/3$ and $N = 169$.

The designed filters should have as much attenuation as possible in the stopband and as little ripple as possible in the passband. Analysing the transfer functions in Figures 8a and 8b, we can see that, for a given w_0, a larger δ (that is, a larger transition, or 'don't care', band), allows a smaller ripple in the passband, as well as a larger attenuation in the stopband. On the other hand, more of the irregular component can leak through a larger transition band, yielding a filter that tends to overadjust the seasonality. Consider now a given δ, the bigger the w_0, the smaller the ripple in the passband (see Figures 8a, 8c, Figure 8b and 8d), but the attenuation in the stopband gets worse.

Being aware that different values of the filter parameters result in distinct transfer functions, it is important to have a methodology for choosing their values. In this work, we adopted the strategy of analysing the performance of the filter when applied to artificial series with behaviour similar to the one of real macroeconomic series. This issue will be dealt with in the following subsection.

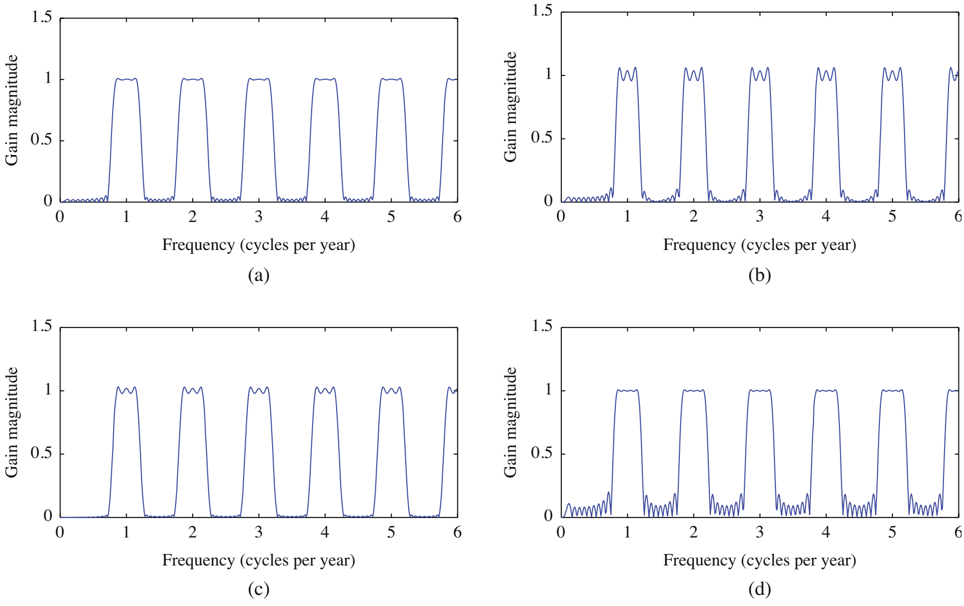

Fig. 8. Magnitude of the transfer function of the S-WLS filter when $N = 169$, $\alpha = 1/3$ and (a) $\delta = 1/10$ and $w_0 = 1$. (b) $\delta = 1/100$ and $w_0 = 1$. (c) $\delta = 1/10$ and $w_0 = 0.3$. (d) $\delta = 1/100$ and $w_0 = 10$.

3.2. The Choice of the S-WLS Filter Parameters

As mentioned at the end of last subsection, we choose the filter parameters by analysing their performance when filtering simulated series. These artificial series should have features similar to the ones of real macroeconomic series. Therefore, our first step was to analyse real macroeconomic data, for the purpose of identifying the behaviour of their moving seasonality. From 144 monthly macroeconomic time series analysed, 53% showed changing seasonal behaviour, according to the F-test for moving seasonality implemented in X-13A-S program, considering a p-value $< 5\%$. Those series are listed in SMT-UFRJ (2014), and were obtained from the OECD (2014), the U.S. Census Bureau (2014), the U.S. Bureau of Labour Statistics (2014), the IPEA (2014), and the IBGE (2014).

An example of a time series with moving seasonality is the 'USA Employment Level' (p-value $< 0.1\%$), from the U.S. Bureau of Labour Statistics (jan/93 to sep/2013). Its seasonal component, adjusted using X-13A-S program, is shown in the plot in Figure 9.

As can be observed from Figure 9, this seasonal component changes its amplitude and shape over the months, confirming the changing seasonality. In order to generate monthly artificial seasonal components with similar behaviour, we used a sinusoidal series whose amplitude is modulated by another sine wave, as follows:

$$S(t) = A\left[1 + b\sin\left(2\pi\frac{t}{k}\right)\right]\left[\cos\left(2\pi\frac{t}{12}\right)\right] \tag{23}$$

where: A is the seasonal amplitude ($A \in \mathbb{R}$); b is related to the rate of change in the signal amplitude ($b \in (0,1)$), k is related to the change in the seasonal pattern, and t is the time index ($t = 1, 2, \ldots$).

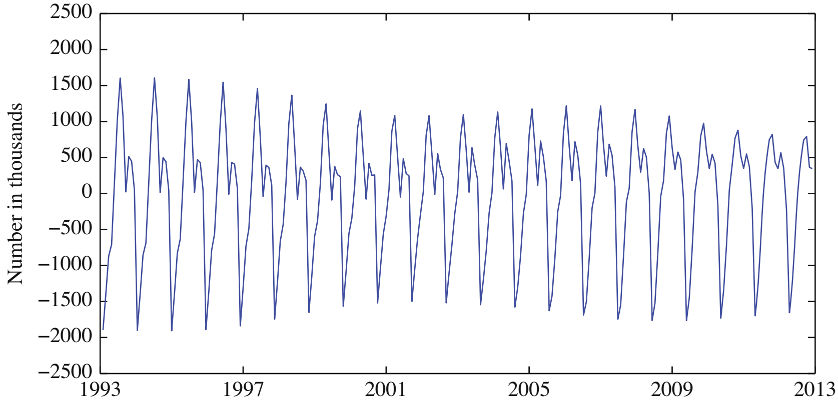

Fig. 9. Seasonal component of the 'USA Employment Level' (jan/93 to sep/2013).

A seasonal component represented by Equation (23) is illustrated in the time and frequency domains in Figures 10a and 10b. Its parameters are: $A = 1350$, $b = 20\%$ and $k = 144$. In Figure 10a it is possible to identify the amplitude change of the seasonal component, and in Figure 10b we notice that the variations in seasonality appear as two sinusoidal components distant $\pm 2\pi/k$ rad/month from the seasonal frequency, that is $2\pi/12$ rad/month. Note that the α parameter of the filter (see Figure 6) must be such that it can properly deal with seasonal frequency variations, that is

$$\alpha \frac{2\pi}{N_s} \geq \frac{4\pi}{k} \tag{24}$$

where N_s is the the seasonal period, that is twelve for monthly data and four for quarterly data.

It is important to note that the parameter α of the filter gives an upper bound to the maximum variation of the seasonal frequency that the filter is able to handle. However, since the filter design method is deterministic, in a practical application one could perform a spectral analysis of the time series prior to the seasonal adjustment in order to estimate

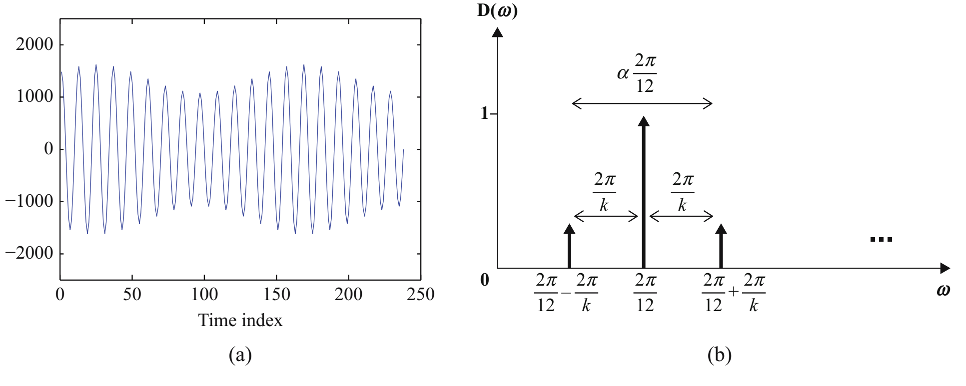

Fig. 10. Seasonal component (Equation (23)) (a) in time domain, (b) in frequency domain.

the bandwidth of the seasonal variation. With this estimate, one could design a filter that would have the α parameter large enough to handle the amount of seasonal variation present in the time series.

An artificial seasonal signal such as the one in Equation (23) can be used to evaluate the response of the filter for different levels of variation in seasonality, either in amplitude or frequency.

To determine the appropriate α for the filter, we analysed the seasonal component of a wide range of macroeconomic time series with moving seasonality. This analysis led us to $\alpha = 1/3$ as a good compromise, that is appropriate for most of the analysed series (in time domain, $\alpha = 1/3$ corresponds approximately to a range between ten and 14 months). Note that, in order to accommodate as much variation on the seasonal component as possible, α should be as large as possible. However, we cannot increase α too much, because the larger the α, the larger the leak of the irregular component through the passbands around the seasonal components, which increases the error in seasonality estimation.

After this, we found a good combination of the parameters δ and w_0, based on the seasonal adjustment of artificial data containing moving seasonal behaviour (Equation (23)), trend and irregular components with the same features of the real time series listed in the webpage in SMT-UFRJ (2014). As observed in Figure 8, for a given α, we have to vary δ and w_0 to find a good compromise between the attenuation in the stopband and the ripple in the passband. In other words, these parameters are responsible, respectively, for the error in the seasonality estimation for the noiseless case (no irregular component) and for the error due to the irregular component, as discussed in Subsection 3.1.

The results showed that a good compromise for the parameters α, δ and w_0 is given by $\alpha = 1/3$, $\delta = 1/30$ and $w_0 = 1$. The complete methodology used to find the combination of the filter parameters is exposed in Appendix B.

Regarding the choice of filter length, it is important to note that one of the aims of this article is to compare the S-WLS filter performance with the one of X-11, considering the filters in the automatic option. Therefore, we only compared the performances of S-WLS and X-11 for the same filter lengths.

Figure 11 shows the transfer function of the S-WLS filter together with the one of the X-11 filter of the same length ($N = 145$). The dashed line represents the transfer function

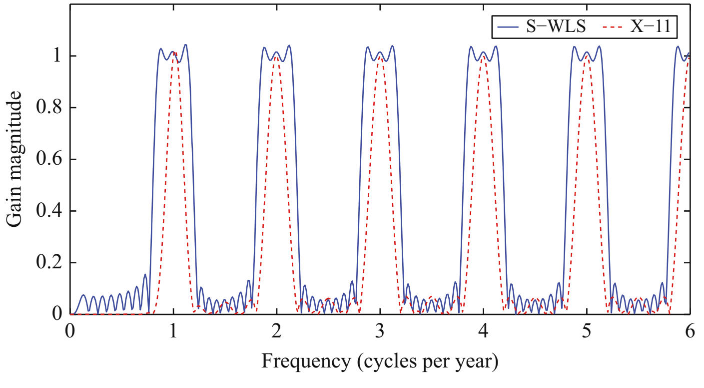

Fig. 11. Magnitude of the transfer function of the S-WLS filter and X-11 filter.

of the X-11 filter, and the continuous line represents the transfer function of the proposed S-WLS filter. As can be seen, the bandwidth of the proposed filter around the seasonal frequencies is larger than the one of X-11, with just a moderate amount of ripple. This allows estimating the seasonal component more accurately in the presence of instability in the seasonal frequency. Yet, the attenuation at the stopband is equivalent to the one of X-11, thus keeping the leaks of the irregular component at a level similar to the one of X-11.

It is important to highlight that, although in this article we determined the design parameters based on the behaviour of a large amount of time series, this filter can be designed according to the characteristics of a specific time series.

The MATLAB program used to implement the S-WLS filter is provided in SMT-UFRJ (2014). In the next section we will present a summary of the experimental results obtained with the S-WLS filter.

4. Results: the S-WLS Filter Performance

Since real time series have unobserved components, we decided to use artificial ones in order to better assess the filter performance. This is so because all the parameters of an artificial series are known and the estimation errors of the seasonal adjustment method can be precisely computed.

The artificial time series used were generated with several degrees of moving seasonality, considering some seasonal behaviours that, in aggregate, characterise the variety of monthly macroeconomic series (see the data in SMT-UFRJ (2014)). Their generation procedure is fully described in the next subsection.

To identify in which conditions of moving seasonality the proposed S-WLS filter performs better than the X-11 method, we applied both S-WLS and X-11 filters to seasonally adjust the mentioned series.

4.1. Data: Application on Artificial Time Series

To assess the ability of S-WLS filter to provide a satisfactory seasonal adjustment for series with moving seasonality, as well as to determine the conditions in which this filter performs better that X-11, we used monthly artificial time series with additive decomposition. These series were divided into two sets: in the first set the series were composed of a seasonal component with moving seasonality added to an irregular component; in the second set of data, a trend component following a cubic polynomial was added to these series, so that the performance of the proposed filters in the presence of a trend component could be assessed. We have chosen an order three polynomial to allow a fair comparison with the X-11 method. This is so because X-11 uses the Henderson filters, which can handle polynomial trend up to order three.

The seasonal component was generated with three parameters (A, b, k) defined in Equation (23). The choice of the parameters A, b, k, as well as the standard deviation of the irregular (s), was based on the characteristics of real monthly series, as mentioned before. As an example, in Figure 12a we show an artificial series with moving seasonality, where we have a cubic polynomial trend component added to an irregular component with parameters $b = 40\%$, $k = 120$, and $A/s = 6$ (Equation (23)). In Figure 12b we show this series without the trend component, and Figure 12c shows just the seasonal component.

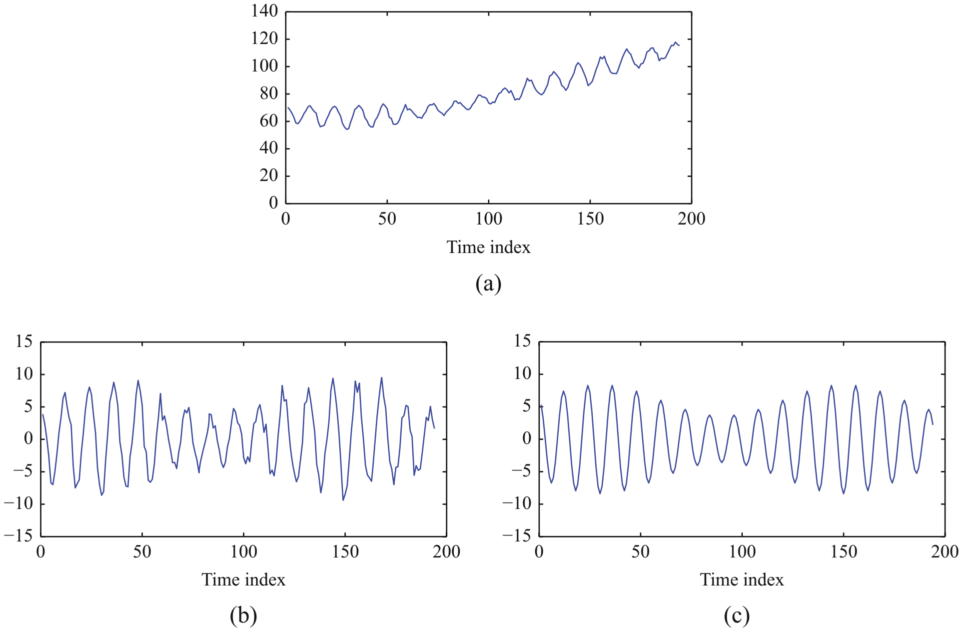

Fig. 12. Equation (23) with $b = 40\%$, $k = 120$, and $A/s = 6$ (a) trend plus seasonal component and irregular, (b) seasonal component plus irregular, (c) seasonal component.

In this analysis we used time series of size 400. It is important to note that it lies outside the scope of this work to extend the series using forecast and backcast. We have done so to avoid masking the differences among the analysed filters. Therefore, since the considered filters (S-WLS and X-11) are symmetric, observations at both ends of the series had to be discarded. It is also important to note that these series were generated without outliers or missing values, so we could focus just on the filters, suitability to extract seasonality.

We generated 1,200 time series. Each simulation was replicated 100 times, randomising the irregular component.

4.2. *Criteria for Comparison*

To assess the ability of S-WLS filter to provide a satisfactory seasonal adjustment for series with moving seasonality, as well as to determine the conditions in which this filter performs better that X-11, we compared the accuracy in seasonality estimation of both filters when applied to the artificial time series. The procedure used in this performance comparison is described by the following steps:

(1) initially, the artificial series were seasonally adjusted by X-11 method considering the seasonal moving averages and Henderson trend filters in the automatic procedure of X-13A-S program;
(2) the X-11 filter that showed the best SNR for the analysed series was chosen and the filter length was determined;
(3) after determining the filter length, the proposed S-WLS filter with the same length was applied to the data.

The accuracy of each set of estimates (S-WLS and X-11) was measured by comparing them to the known seasonal component underlying that series. For this we used the signal-to-noise ratio (SNR – see details in Appendix C), the Mean Squared Error (MSE) and the Mean Absolute Deviation (MAD). We define the mean of the MSE and the mean of the MAD as the average of these statistics over 100 replications of the irregular. A one-sided t-test was applied to the pairs of means of the MSE statistics obtained for S-WLS and X-11 filters, with the alternative hypothesis $\mu_{S-WLS} < \mu_{X-11}$ (negative difference). The same was done considering the MAD statistic.

4.3. Simulation Results

In order to evaluate the conditions under which the S-WLS filters have a better performance than X-11, we considered different possibilities for the seasonal component. These characteristics refer to the parameters b, k, and A (Equation (23)), which were taken from macroeconomic series.

The parameter b is related to the rate of change in the seasonal amplitude, taking values in the interval (0,1). In real data the maximum value found for b was 52%.

Table 3 shows a performance comparison for values of b from 10% to 80% considering $A/s = 6$ and $k = 120$. Figure 13a illustrates the relation between the MSE of X-11 and of the S-WLS filter.

As can be seen, the higher the b value, the better the MSE of the S-WLS filter is compared to the one of X-11. Note that the MSE of the S-WLS filter does not change substantially with the variation of b, while the MSE of X-11 significantly changes with b. The same occurs with the MAD and SNR statistics. Table 3 shows that for smaller values of b, the performance of X-11 tends to improve relative to the one of S-WLS. It is important to note that the value of b from which S-WLS starts to perform better than X-11 depends on the values of k and A/s.

In order to evaluate the performance of the filters S-WLS and X-11 based on the variation of k, we set $b = 40\%$ and $A/s = 6$. As the parameter k is related to the change in the seasonal pattern, the smaller the k, the more unstable the seasonality. In these cases, S-WLS tends to perform better than X-11. Table 4 shows numerical figures illustrating this behaviour.

Figure 13b shows that when k decreases, the MSE of the S-WLS filter remains at the same level, indicating robustness of this filter, while the MSE of X-11 increases. Considering that the S-WLS filter uses the parameter $\alpha = 1/3$ (Subsection 3.2), the minimum value for k that it is able to deal with is 72.

The ratio between the amplitude of the signal (A) and the standard deviation of the irregular component (s) has a substantial influence on the MSE of the filters, as can be seen in Figure 13c. In both filters (X-11 and S-WLS), the MSE drops as the ratio A/s increases, but in the S-WLS this drop is more pronounced. Figure 13c and Table 5 also show that for large values of A/s, the S-WLS outperforms the X-11 filter. In typical series, the minimum value observed of A/s was 2.2 and the maximum was 11.7, and 50% of the monthly series with additive decomposition showed $A/s \geq 6$.

It is important to mention that for different values of k and b, the ratio A/s in which the proposed filter outperforms X-11 changes. For $k = 120$, the minimum A/s for which

Table 3. MSE, MAD, and SNR for b values, with A/s = 6, k = 120: monthly data with trend component (Equation (23)).

	MSE			MAD			SNR		
b	S-WLS	X-11	*p*-value	S-WLS	X-11	*p*-value	S-WLS	X-11	S-WLS/X-11
10%	0.940	0.463	1	0.773	0.547	1	53.7	110.1	0.5
15%	0.990	0.648	1	0.792	0.643	1	50.5	77.1	0.7
20%	1.002	0.697	1	0.798	0.665	1	50.9	71.8	0.7
25%	0.988	0.769	1	0.794	0.702	1	51.4	66.0	0.8
30%	0.992	0.850	1	0.792	0.740	1	52.0	60.3	0.9
40%	1.010	1.065	0.001	0.801	0.833	0.000	53.5	49.7	1.1
50%	1.006	1.331	0.000	0.800	0.940	0.000	55.5	41.1	1.3
60%	1.005	1.636	0.000	0.800	1.050	0.000	57.8	34.5	1.7
70%	1.022	2.012	0.000	0.808	1.173	0.000	60.5	29.4	2.1
80%	1.022	2.494	0.000	0.805	1.312	0.000	63.6	25.5	2.5

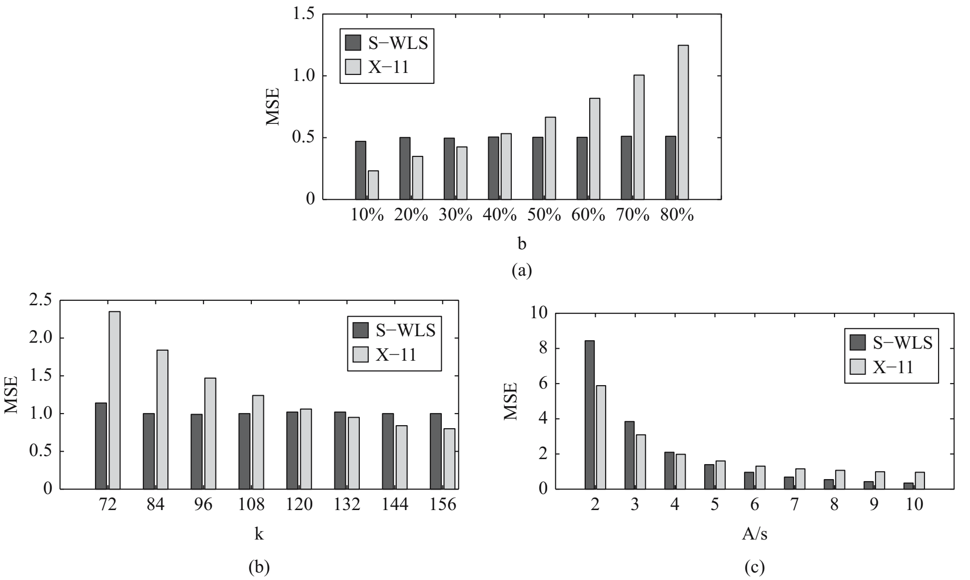

Fig. 13. Average of the MSE statistic in the simulations: X-11 and S-WLS filter (a) values of b, with A/s = 6, k = 120. (b) values of k, with A/s = 6, b = 40%. (c) values of A/s, with b = 40%, k = 120 (Equation (23)).

S-WLS overperforms X-11 is five; however, if $b = 40\%$ and $k = 72$, then $A/s \geq 3$ is enough for the S-WLS filter to perform better that X-11. The complete table with all the possibilities is available in SMT-UFRJ (2014).

Another way to verify the adequacy of the seasonal adjustment filter is to analyse the spectrum of the deseasonalised series. In Figures 14a and 14b we show the spectrum of the series deseasonalised by S-WLS filter and by X-11, respectively. The parameters of the series were the same as those used in the examples above: $b = 40\%$, $k = 120$ and $A/s = 6$. In Figure 14a it is possible to note that there is a peak in the frequency 1/12 cycles/month, indicating that some seasonality remains in the series after being deseasonalised by X-11, while in Figure 14b there is no peak, meaning that the seasonality was removed after applying the S-WLS filter to the data.

The performances of the S-WLS filter and the X-11 are analysed and compared for artificial time series based on Equation (3), that simulates a seasonal component with nonstationary changes. The results are presented in Table 6, showing the MSE and MAD for all combinations of Henderson filter and seasonal moving average filters.

Analysing the results we note that the proposed method (S-WLS) is able to estimate this kind of non-stationary seasonality better than X-11.

4.4. Data: Application on Real Time Series

In order to illustrate the S-WLS filter on a real-life time series, we applied it to the Austrian Consumer Price Index (all items non-food non-energy). This monthly time series was obtained from the OECD (http://stats.oecd.org/index.aspx?DatasetCode=MEI, extracted on April 2016), with a time span of 41 years (from jan/1975 to jan/2016). Besides this,

Table 4. MSE, MAD, and SNR for k values, with A/s = 6, b = 40%: monthly data with trend component (Equation (23)).

	MSE			MAD			SNR		
k	S-WLS	X-11	p-value	S-WLS	X-11	p-value	S-WLS	X-11	S-WLS/X-11
72	1.14	2.35	0.000	0.85	1.25	0.000	48.6	23.3	2.1
84	1.00	1.84	0.000	0.80	1.11	0.000	53.5	30.2	1.8
96	0.99	1.47	0.000	0.79	0.99	0.000	53.5	35.9	1.5
108	1.00	1.24	0.000	0.80	0.90	0.000	53.5	42.7	1.3
120	1.02	1.06	0.001	0.80	0.83	0.000	53.5	49.7	1.1
132	1.02	0.95	1	0.81	0.79	0.999	53.6	55.8	1.0
144	1.00	0.84	1	0.80	0.74	1	53.7	64.5	0.8
156	1.00	0.80	1	0.80	0.71	1	53.8	65.8	0.8
180	1.01	0.73	1	0.80	0.68	1	53.8	73.4	0.7

Table 5. MSE, MAD, and SNR for A/s values, with b = 40% and k = 120: monthly data with trend component (Equation (23)).

A/s	MSE			MAD			SNR		
	S-WLS	X-11	p-value	S-WLS	X-11	p-value	S-WLS	X-11	S-WLS/X-11
2	8.438	5.884	1	2.320	1.939	1	6.41	9.17	0.70
3	3.845	3.089	1	1.565	1.404	1	14.38	17.76	0.81
4	2.096	1.981	0.999	1.153	1.130	0.977	25.47	26.42	0.96
5	1.394	1.603	0.000	0.945	1.025	0.000	39.62	34.13	1.16
6	0.957	1.308	0.000	0.780	0.932	0.000	56.74	40.56	1.40
7	0.686	1.156	0.000	0.661	0.884	0.000	76.59	45.72	1.68
8	0.540	1.070	0.000	0.587	0.855	0.000	99.50	49.90	1.99
9	0.427	0.991	0.000	0.520	0.827	0.000	125.14	53.23	2.35
10	0.347	0.960	0.000	0.468	0.817	0.000	152.80	55.86	2.74

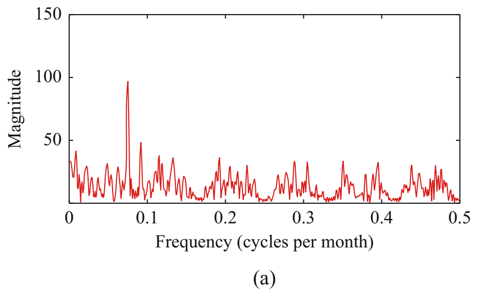

Fig. 14. Spectrum of the deseasonalised series (a) by X-11. (b) by S-WLS.

we used X-11 (X-13A-S program) to seasonally adjust this time series. Then, we compared the results.

Note that we do not perform series extrapolation at its extremes because the effects of the asymmetrical weights could mask the differences that we want to analyse.

The seasonal component extracted by X-11 and S-WLS filters is shown in Figures 15a and 15b, respectively.

We can see that by 1994 the seasonal component of S-WLS is higher than the one of X-11, showing that it can capture more seasonality than X-11.

5. Concluding Remarks

In this article we have proposed a seasonal adjustment filter design methodology, in which the main feature is to be robust to changes in the seasonal behaviour. This filter, named S-WLS, was designed in the frequency domain, based on least squares criteria, allowing the specification of an adequate passband width for filtering series with moving seasonality.

Several seasonal adjustment filters have been proposed in the literature. Our contribution is in the fact that this robustness is achieved while preserving its automatic character (so, it can be used to adjust a large amount of series). In addition, its parameters were determined based on the seasonal behaviour of typical macroeconomic series.

Table 6. Comparison results of S-WLS and X-11 for different Henderson filters and seasonal moving average filters.

		MSE		MAD	
Henderson filter	Seasonal MA	S-WLS	X-11	S-WLS	X-11
9	3 × 3	0.021	0.066	0.097	0.198
9	3 × 5	0.019	0.090	0.091	0.230
9	3 × 9	0.021	0.148	0.097	0.298
13	3 × 3	0.019	0.063	0.092	0.193
13	3 × 5	0.020	0.088	0.094	0.228
13	3 × 9	0.020	0.149	0.096	0.299
23	3 × 3	0.019	0.049	0.092	0.167
23	3 × 5	0.020	0.080	0.094	0.213
23	3 × 9	0.020	0.151	0.096	0.299

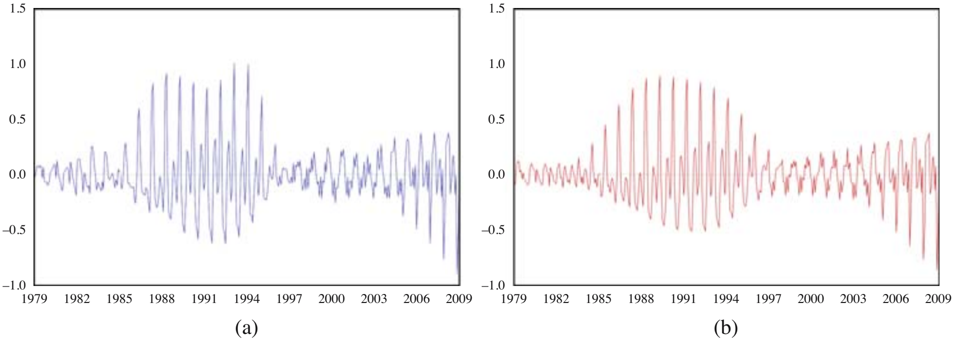

Fig. 15. Seasonal component of Austrian Consumer Price Index (all items non-food non-energy) (a) extracted with X-11. (b) extracted with S-WLS.

With the aim of assessing the performance of our filter, we compared it to the X-11 method, since this is an ad-hoc filter and has been widely used. In the comparisons we took care to use the S-WLS filters with the same lengths of X-11 (considering the automatic procedure in X-13A-S). These comparisons were performed both using time domain characteristics, based on the statistics MSE and MAD, and frequency domain characteristics, using the SNR and the inspection of the spectrum of the seasonally adjusted series. In these comparisons, we used simulated monthly data with changing seasonal patterns based on the ones of the typical macroeconomic series.

Due to space constraints, this study was limited to monthly additive series. Yet, we have verified that our filter can be easily extended to quarterly data and other periodicities, and it also performs well in multiplicative series.

The simulation results show that for series with a very slowly changing seasonal pattern, our filter tends to overadjust the data in comparison to X-11. On the other hand, as the degree of moving seasonality increases, X-11 tends to underadjust the series (i.e., not to remove all the seasonality), while our filter shows a good performance. This occurs thanks to the larger passband width of the S-WLS filter that allows robustness in cases of moving seasonality, providing a better quality of adjustment than X-11.

Detailing these findings regarding the characteristics of the seasonal component, we have that:

(i) if the amplitude of the seasonal component is large when compared to the standard deviation of the irregular component, the proposed S-WLS filter performs better than X-11;
(ii) the same occurs when the rate of change in the seasonality amplitude is large enough;
(iii) regarding the period of change in the seasonal pattern, the faster the changes in seasonal pattern, the better is the performance of S-WLS.

In brief, we recommend the X-11 method – in X-13A-S – if the variations in seasonality are sufficiently small. In cases of stronger variations, the proposed S-WLS filter performs better. Interesting further investigation would be to extend the current work for the case of multivariate series, such as is done in Infante et al. (2015) in the context of testing common seasonal patterns.

Appendix A. The X-11 Algorithm in the Frequency Domain

First of all, we define the $P \times Q$ moving average in the frequency domain, using the Z-transform, as follows:

$$M_{PXQ}(z) = \frac{z^{\left(\frac{p+Q-2}{2}\right)}}{PQ}\left[\sum_{n=0}^{P-1}(n+1)z^{-n} + \sum_{n=P}^{Q-2}Pz^{-n} + \sum_{n=Q-1}^{P+Q-1}(P+Q-1-n)z^{-n}\right] \quad \text{(A.1)}$$

The standard X-11 algorithm as shown in (Dagum 1988) and in (Findley et al. 1998) is presented below. For this, we consider a monthly time series and additive decomposition: $y_t = t_t + s_t + i_t$, where y_t is the original series, and t_t, s_t, i_t are the non-observable components of trend, seasonality and irregular, respectively. The filtering operations are presented in the frequency domain, using the Z-transform.

Stage 1 Preliminary Estimates

The first estimate of the trend component is obtained by applying a ‘centered 12-term’ moving average $M_{2X12}(z)$, that is

$$T^{(1)}(z) = Y(z)M_{2X12}(z). \quad \text{(A.2)}$$

The first estimate of Seasonal and Irregular components (SI) is given by

$$SI^{(1)}(z) = Y(z) - T^{(1)}(z). \quad \text{(A.3)}$$

One obtains the preliminary estimate of the Seasonal Factor $\hat{S}^{(1)}(z)$ by applying a weighted five-term moving average ($M_{3X3}(z^{12})$) to the SI component,

$$\hat{S}^{(1)}(z) = SI^{(1)}(z)M_{3X3}(z^{12}). \quad \text{(A.4)}$$

Then, the initial Seasonal Factor ($S^{(1)}(z)$) and the preliminary Seasonal Adjustment ($A^{(1)}(z)$) are obtained as

$$S^{(1)}(z) = \hat{S}^{(1)}(z) - \hat{S}^{(1)}(z)M_{2X12}(z) \quad \text{(A.5)}$$

$$A^{(1)}(z) = Y(z) - S^{(1)}(z). \quad \text{(A.6)}$$

Stage 2 Seasonal Factors and Seasonal Adjustment

We then perform the intermediate trend estimate by applying the ‘13’-term Henderson filter ($H_{13}(z)$) to the seasonally adjusted series, from Equation (A.6),

$$T^{(2)}(z) = A^{(1)}(z)H_{13}(z). \quad \text{(A.7)}$$

The second estimate of the SI component is then given by

$$SI^{(2)}(z) = Y(z) - T^{(2)}(z). \tag{A.8}$$

We then obtain the second estimate of the Seasonal Factor via a seven-term moving average ('3 × 5' seasonal moving average):

$$\hat{S}(2)(z) = SI^{(2)}(z)M_{3X5}(z^{12}) \tag{A.9}$$

from which the Seasonal Factor ($S^{(2)}(z)$) and the Seasonally Adjusted series ($A^{(2)}(z)$) are obtained as

$$S^{(2)}(z) = \hat{S}^{(2)}(z) - \hat{S}^{(2)}(z)M_{2X12}(z) \tag{A.10}$$

$$A^{(2)}(z) = Y(z) - S^{(2)}(z). \tag{A.11}$$

As the operations presented in Stages 1 and 2 are linear, it is possible to represent them as an equivalent filter of X-11 method for the seasonal adjustment. This filter, from Equations (A.2) to (A.11), is

$$\begin{aligned} A^{(2)} = Y(z)\{1 - \mathrm{M}_{3\times5}(z^{12})[1 - \mathrm{M}_{2\times12}(z)][1 - H_{13}(z) \\ \times\{1 - [1 - \mathrm{M}_{2\times12}(z)]^2\mathrm{M}_{3\times3}(z^{12})\}]\}. \end{aligned} \tag{A.12}$$

Appendix B. The Procedure for Selecting the Values of the S-WLS Filter Parameters

The procedure for selecting the values of the S-WLS filter parameters is summarised in the following steps below. For further details, the reader is referred to SMT-UFRJ (2014):

B.1. Experimental Determination of the Values For A, b, k (see Equation (23)), and the Standard Deviation (s) of the Irregular Component

For each real time series, we used the X-13A-S program (considering the X-11 adjustment mode) to estimate the seasonal component. The behaviour of the seasonal components of these real time series with moving seasonality was individually analysed. The specification of the value of α was based on the values of k, using the relation in Equation (24) exemplified by Figure (10).

B.2. Choosing the Filter Length

The X-11 filter that best fitted the data was chosen based on the theoretical evaluation of the SNR, for all the combinations of seasonal moving average filters and Henderson filters in the automatic mode of X-11. For this we used the values of A, b, k and s from step A.1 to build artificial seasonal signals and irregular components. Then we searched for the best X-11 filter for a given combination of these parameters, based on the X-11 SNR (Equation (C.9)).

B.3. Searching for the S-WLS Parameters

Based on the filter length defined in Subsection 2, we searched for the combination of δ and w_0 with largest ratio between the SNRs of S-WLS and X-11 (see Equations (C.8) and (C.9)). At this stage, we restricted the choice of the lengths of the S-WLS filter to the possible lengths of the X-11 filter. The parameter α was fixed at 1/3, while δ and w_0 values were varied over a wide range. Then, for several combinations of A, k, b (Equation (23)) and s, we chose the S-WLS filters that were, in general, based on SNR better than X-11. Since there is no single set of parameters that best fits the data, we worked with the top eight combinations.

B.4. Determination of the Best Parameters for a Wide Range of the Parameter k

At this stage we performed a simulation by filtering time series with moving seasonality, cubic trend component, and irregular component following a $N(0,\sigma^2)$. In it, 100 replications of the irregular component were generated for each series. The artificial seasonal components were created considering 100 values of k (Equation (23)), drawn randomly from a set of possible values based on the seasonal behaviour of macroeconomic data.

To search for the S-WLS parameters, we first chose the X-11 filter that had the lowest MSE for the estimation of the seasonal component of each replication of the series. Setting the same length of the S-WLS filter as the one of X-11, we searched for the parameters δ and w_0 that provided good MSE figures.

The S-WLS parameter combination that obtained, in general, the lowest MSE compared to the MSE of the X-11, was identified as $\alpha = 1/3$, $\delta = 1/30$ and $w_0 = 1$.

Appendix C. X-11 and S-WLS SNRs

When we filter a time series to extract its irregular component, according to the model in Equation (2), the errors at its output may have three main causes:

(i) residuals of the trend component;
(ii) irregular component at the output of the filter;
(iii) errors caused by the seasonal component.

In our case, the errors in (i) are automatically eliminated by the filter's structure (Equation (8)). In Sections B.1 and B.2, we deal with the errors in (ii) and (iii), respectively.

C.1. Variance of the Irregular at the Output of the Filter (Noise Power)

When a stochastic process $x(t)$ with power spectral density $S_X(e^{i\omega})$ is input to a filter with transfer function $\mathcal{H}(e^{i\omega})$, the power spectral density of its output $y(t)$ is $S_X(\omega)|\mathcal{H}(e^{i\omega})|^2$ (Diniz et al. 2010). Since an uncorrelated irregular component with variance, or noise power, σ_X^2 has a power spectral density equal to σ_X^2, then the power spectral density of its output $y(t)$ is

$$S_Y(e^{i\omega}) = \sigma_X^2|\mathcal{H}(e^{i\omega})|^2. \quad \text{(C.1)}$$

Therefore, the variance of the irregular component, or noise power, of the output of the filter is

$$\sigma_Y^2 = \int_{-\pi}^{\pi} S_Y(e^{i\omega})d\omega = \int_{-\pi}^{\pi} \sigma_X^2 |\mathcal{H}(e^{i\omega})|^2 d\omega = \sigma_X^2 \sum_{t=-\infty}^{\infty} |h(t)|^2. \qquad \text{(C.2)}$$

where the rightmost equality comes from Parseval's theorem (Diniz et al. 2010), with $h(t)$ being the filter coefficients. In other words, in the case of an uncorrelated irregular component, the noise power of the irregular at the output of a filter is proportional to the sum of the squares of its coefficients.

C.2. Errors Caused by the Seasonal Component

(a) The case of the S-WLS filter
Consider Figure C.16, which illustrates a typical frequency response of the S-WLS filter at the passband (see also Figures 8a to 8d in Subsection 3.1). There, we highlight two important deviations from the desired unit gain in the passband. The first one is given by γ_0, which is the gain at the fundamental frequency. The second one gives the maximum deviation from the desired response at the passband. Since the desired response is 1, and the corresponding gain is γ_1, the maximum deviation is given by $|1 - \gamma_1|$.

For an input time series according to Equation (23), since it is composed of sinusoidal components, the contribution of the above deviations to the noise power is,

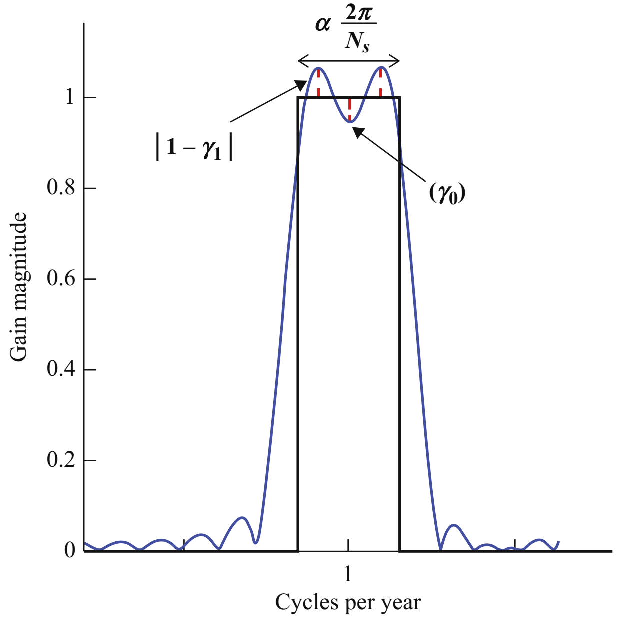

Fig. C.16. Deviations from the desired passband gain for the S-WLS filter.

in the worst case,

$$e_1 = (1-\gamma_0)^2 \frac{A^2}{2} + (1-\gamma_1)^2 \frac{A^2 b^2}{4}. \tag{C.3}$$

where A is the seasonal amplitude ($A \in \mathbb{R}$), b is related to the rate of change in the seasonal amplitude ($b \in (0,1)$), k is related to the change in the seasonal pattern, and t is the time index ($t = 1, 2, \ldots$).

(b) The case of the X-11 filter

As can be seen from the frequency responses of the X-11 filter in Figure 11, Subsection 3.2, the typical frequency response of the X-11 filter always has a peak at the seasonal frequency (1/12 cycles per month, for monthly series, and 1/4 cycles per quarter, for quarterly series). As you move away from the peak, the response decreases monotonically. Therefore, the two largest deviations from the ideal unit gain in the passband are given by the two frequencies at the edges of the passband, as illustrated in Figure C.17, in which the response function differs from the 'ideal' by $(1-\beta_1)$ and $(1-\beta_2)$. There, the dashed line represents the spectrum of the X-11 equivalent filter, and the continuous line represents the magnitude of the ideal frequency response for an allowed seasonal frequency variation of $\alpha(2\pi/N_s)$ around the nominal frequency (in this case, $2\pi/N_s$ or its harmonics).

Using an argument equivalent to the one that led to Equation (C.3), the contribution of the above deviations to the noise power is, for the X-11 filter,

$$e_2 = \frac{A^2 b^2}{4} \frac{[(1-\beta_1)^2 + (1-\beta_2)^2]}{2} \tag{C.4}$$

(c) Computation of the SNR of the S-WLS and X-11 filters

Therefore, if we refer to the sum of the squares of the coefficients of the S-WLS filter as SQ and to the one of the X-11 as S, then we have, from Equations (C.3) and (C.4),

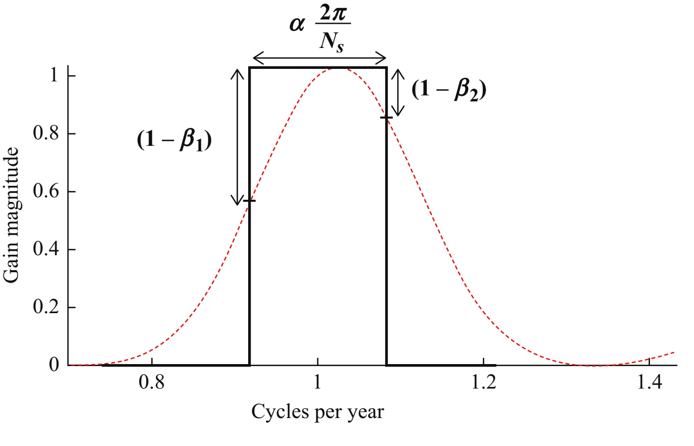

Fig. C.17. Deviations from the desired passband gain for the X-11 filter.

that the total noise power at the output of S-WLS and X-11 filters is given by:

$$e_{S-WLS} = (1-\gamma_0)^2\frac{A^2}{2} + (1-\gamma_1)^2\frac{A^2b^2}{4} + SQ\,\sigma^2 \tag{C.5}$$

$$e_{X-11} = \frac{A^2b^2}{8}[(1-\beta_1)^2 + (1-\beta_2)^2] + S\,\sigma^2. \tag{C.6}$$

Since the seasonal signal in Equation (23) is composed of three sinusoids, its average squared value is given by

$$E_s = A^2\frac{1}{2} + \frac{A^2b^2}{4}\frac{1}{2} + \frac{A^2b^2}{4}\frac{1}{2} = \frac{A^2}{2}\left(1+\frac{b^2}{2}\right). \tag{C.7}$$

Therefore, from Equation (C.5) to (C.7), we have that the SNRs of the S-WLS and X-11 filters are:

$$SNR_{S-WLS} = \frac{\frac{A^2}{2}\left(1+\frac{b^2}{2}\right)}{(1-\gamma_0)^2\frac{A^2}{2} + (1-\gamma_1)^2\frac{A^2b^2}{4} + SQ\,\sigma^2} \tag{C.8}$$

$$SNR_{X-11} = \frac{\frac{A^2}{2}\left(1+\frac{b^2}{2}\right)}{\frac{A^2b^2}{8}[(1-\beta_1)^2 + (1-\beta_2)^2] + S\,\sigma^2}. \tag{C.9}$$

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