

# ON THE EMPIRICAL RATE-DISTORTION PERFORMANCE OF COMPRESSIVE SENSING

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## ABSTRACT

Compressive Sensing (CS) is a new paradigm in signal acquisition and compression that has been attracting the interest of the signal compression community. When it comes to image compression applications, it is relevant to estimate the number of bits required to reach a specific image quality. Although several theoretical results regarding the rate-distortion performance of CS have been published recently, there are not many practical image compression results available. The main goal of this paper is to carry out an empirical analysis of the rate-distortion performance of CS in image compression. We analyze issues such as the minimization algorithm used and the transform employed, as well as the trade-off between number of measurements and quantization error. From the experimental results obtained we highlight the potential and limitations of CS when compared to traditional image compression methods.

**Index Terms**— Compressive Sensing, Rate-Distortion Analysis, Quantization, Image Coding.

## 1. INTRODUCTION

Ordinary images, as well as most natural and manmade signals, are compressible and can, therefore, be well represented in a domain in which the signal is sparse. Standard image acquisition techniques follow the *sample-then-compress* framework. This involves sampling at a large rate only to discard most of the acquired information using a compression scheme that exploits the sparse representation.

In this context, Compressive Sensing (CS) comes out as a new paradigm for data acquisition; it gives a stable and robust algorithm that allows sensing at rates much smaller than the Nyquist limit, while recovering the signals with little corruption [1, 2]. Reconstruction from the measurements is achieved by convex optimization techniques (e.g.  $l_1$ -norm minimization).

To compress sensed information, it is necessary to make use of quantization schemes that add distortion to the acquired data. Therefore, a relevant contribution to CS theory would be to verify how it performs in the presence of quantization errors and in a rate-distortion sense.

Theoretical results have been established guaranteeing stability of CS to the addition of quantization errors. In [3],

CS encoding of approximately sparse signals with quantized measurements is studied and performance is demonstrated to be within a logarithmic factor to the one of the optimal encoder.

Related works consider strictly sparse signals and evaluate CS when quantization errors are added. In [4], the results of [3] are extended to the scenario where exact sparsity is guaranteed and inefficiencies in terms of rate and performance are verified, suggesting modifications in the uniform scalar quantization method and the reconstruction algorithm. In [5], both of these changes are explored and extensive computer simulations are made confirming their advantages.

The rate-distortion function is used in [6] to compare CS to the ideal compression scheme (where an oracle informs the sparsity pattern) and the loss in performance is evaluated as relatively small (an additive logarithmic rate penalty is observed). In [7], a lower bound on the number of measurements needed to reconstruct a signal is set as a function of the measurements' SNR and rate-distortion function.

However, fundamental questions regarding performance in practical applications still remain unanswered.

In this work we aim at helping to answer some of these questions from an empirical point of view. Though innumerable applications have been suggested, we concentrate our study on the scenario in which CS was first presented and is mostly discussed: image sensing and compression.

It is important to emphasize that it is not within the scope of this work to elaborate further theoretical results or to analyze CS fundamental limitations in idealized scenarios. Rather, we aim at evaluating applications in image acquisition through the use of empirical analysis.

### 1.1. Overview of CS and Basic Notation

Let  $x \in \mathbb{C}^{N \times 1}$  be a vector representation of an image and  $\Psi \in \mathbb{C}^{N \times N}$  a unitary transform that makes  $x$  sparse, i.e.,  $\Psi x = s$ , where  $s$  has only  $S$  nonzero coefficients. Since we are acquiring only  $M \ll N$  measurements, sensing can be denoted by  $y = \Phi x = \Theta s$ , where  $\Phi, \Theta \in \mathbb{C}^{M \times N}$  and  $\Theta = \Phi \Psi^*$ .

CS theory states that it is possible, through the use of a convex optimization algorithm, to recover  $x$  from  $y$  with overwhelming probability if  $\Theta$  satisfies a *Restricted Isometry Property (RIP)* [8]. It has also been shown that this property



**Fig. 1.** Test images.

can be assumed if the entries of  $\Theta$  belong to certain random ensembles and  $M$  is of the same order as  $S \log(N/M)$ .

Moreover, this acquisition technique is robust to sparsity approximations and measurements errors [9]. Let  $y \in \mathbb{C}^{M \times 1}$  be the acquired data corrupted by noise, i.e.,  $y = \Phi x + \eta$ , where  $\|\eta\|_{l_2} \leq \epsilon_q$ . If we reconstruct the signal by solving the convex optimization problem

$$\min_x \|\Psi x\|_{l_1} \quad \text{subject to} \quad \|y - \Phi x\|_{l_2} \leq \epsilon_q. \quad (1)$$

then the recovery error is bounded by the sum of the measurement (quantization) error and the error due to the fact that the signal is not strictly sparse, i.e.

$$\|y - \Phi x\|_{l_2} \leq C \cdot \left( \epsilon_q + \underbrace{S^{-1/2} \|x_S - x\|_{l_1}}_{\epsilon_s} \right), \quad (2)$$

where  $C$  is relatively small and  $x_S$  is an approximation of  $x$  where the  $S$  largest coefficients in the  $\Psi$  domain are observed. This implies that the reconstruction error in CS is of the order of the maximum of the quantization and measurement errors. For details see [9].

## 2. EXPERIMENTAL SETUP

CS investigations were made on four different images of size  $N = 256^2$ , which differ in terms of both sparsity and high energy coefficient distribution in the frequency domain (see Figure 1). Since the images are stored in the computer as a matrix of pixels, we simulate acquisition by means of measurements that involve linear combinations of these pixels.

Measurements were taken by choosing at random  $M$  waveforms of an  $N \times N$  Noiselet transform [10]. Such measurements were chosen because they are highly *incoherent* [1] with the considered sparse domains and the RIP tends to hold for reasonable values of  $M$ . In addition, the matrix created is orthogonal and self-adjoint, thus being easy to manipulate.

The following recovery strategies were considered:

- A. Minimization of the  $l_1$ -norm of the image's DCT (discrete cosine transform);
- B. Minimization of the  $l_1$ -norm of the image's DWT (discrete Wavelet transform);
- C. Minimization of the image's  $TV$  (total-variation) norm; and

- D. Minimization of the  $l_1$ -norm of the image's SVD (singular value decomposition)

The efficiency of each strategy is related to how sparse the images are in the considered domain. The DCT and the Wavelet domains were chosen because of their widespread use in image compression standards. In addition, since most published theorems relate to orthogonal rather than to the more efficient biorthogonal basis, we used an orthonormal Wavelet basis (Coiflet with 2 vanishing moments).

In many recent publications [1, 11], CS researchers have used the total variation (TV) norm, which can be interpreted as the  $l_1$ -norm of the (appropriately discretized) gradient. Applied to images, the TV-norm minimization favors a certain smoothness that is usually found in natural and manmade pictures without penalizing discontinuous features and is, therefore, very effective.

Finally, the SVD was calculated for each image and used to determine sparsity domains because it gives a very accurate sparse representation. This technique requires knowledge of the SVD basis, that is calculated from the whole image information (not available in CS) and requires a large data rate for transmission (which is not taken into account). Nevertheless we used such results as upper bounds that, although loose, give interesting insights into performance limitations.

While strategies A, B and D solve Equation 1, in strategy C the image is reconstructed by solving the following convex optimization problem:

$$\hat{x} = \min_x \|x\|_{TV} \quad \text{subject to} \quad \|y - \Phi x\|_{l_2} \leq \epsilon. \quad (3)$$

### 2.1. The Rate-Distortion Curve

Scalar uniform quantization was considered and tests were made for different quantization steps in order to select the best for each compression rate and plot rate-distortion curves.

Rate was calculated as  $(M/N)H_y$ , where  $H_y$  is the entropy (in bits per pixel) of the quantized measured data  $y$  estimated from its histogram. The problem of unused quantization values is resolved by considering each of them to have occurred once.

### 2.2. Implementation Aspects

The experiments were implemented in MATLAB and the *l1-Magic* [12] toolbox was used to solve both optimization problems (Equations 1 and 3). For each image, recovery strategy and quantization step, the parameter  $\epsilon_q$  was chosen experimentally in order to maximize the PSNR (peak signal-to-noise ratio).

The Wavelet basis was generated using the WAVELAB [13] package and the Noiselet basis using an algorithm made available by Romberg [11].

## 3. RESULTS

In Figure 2 the rate-distortion curve was plotted for all tested images and considered strategies. We can observe that CS

recovery schemes that perform the  $l_1$ -norm minimization in the Wavelet domain are far less efficient than the JPEG2000 standard. However, by analyzing the results for strategy D and for the test image *Phantom* on strategy C, we can see that there is room for improvement; in both cases one gets better results than with JPEG2000. The *Phantom* image in the frequency domain and the SVD transform are both very sparse. This indicates that by choosing representations that strengthen sparsity one can reduce not only the number of measurements needed to reconstruct the signal but also the approximation error.

It is important to mention that, though strategy D presents an upper bound to CS performance, it is not really practical because it requires an *a priori* knowledge of the image's SVD. Figure 5 highlights this argument by contrasting recovery of the image *Camera man* using as a basis *Camera man*'s SVD and *Lena*'s SVD.

In Figure 3 the Rate  $\times$  PSNR curve was plotted for strategies B and C using varying quantization steps. It can be observed that for a particular compression rate, each image and recovery strategy has an optimal quantization step that produces the highest PSNR. If the image is not sparse in the considered domain, the curves show that it is more efficient to take a large number of measurements and compensate for the potential rate increase by enlarging the quantization step.

One can also observe that for a fixed PSNR, the ideal quantization step is approximately the same in all evaluated scenarios. This observation is closely related to the result in Equation 2, which indicates that the recovery error is of the same order as the largest of the approximation and measurement errors [3]. The PSNR determines the acceptable distortion and, therefore, the values of  $\epsilon_q$  and  $\epsilon_s$ . While  $\epsilon_q$  only depends on the quantization step,  $\epsilon_s$  depends on the sparsity distribution and, hence, on the number of measurements.

The same comment can be made after observing Figure 4, that shows the results in terms of Number of Measurements  $\times$  PSNR. For each strategy the number of measurements determines  $\epsilon_s$ ; in addition, all quantization steps that make  $\epsilon_q$  of the order of  $\epsilon_s$  (or smaller) result in the same PSNR (see Equation 2). Therefore, all curves overlap until the number of measurements is large enough so that  $\epsilon_s$  exceeds  $\epsilon_q$  (see Figure 4.(b)). In Figure 4.(a), it is noteworthy that for quantization steps smaller than 3, the curves overlap completely. This is so because as the errors due to sparsity are very large, reducing the quantization step is ineffective in increasing PSNR. In contrast, in Figure 4.(c), where the image is strongly sparse in the considered domain (SVD),  $\epsilon_s$  tends to be much smaller, and therefore such behavior is not observed.

A complete list of the results and the MATLAB code that reproduces them is available at [www.impa.br/~aschulz/CS](http://www.impa.br/~aschulz/CS).

#### 4. DISCUSSION AND CONCLUSIONS

The results obtained during this study suggest contexts in which improvements in the CS acquisition strategy could lead to better rate-distortion performance.

It has already been emphasized that sparsity plays a very important role in recovery (e.g. SVD). Therefore, to make CS applications in imaging practical, domains that enhance sparsity, such as biorthogonal Wavelets and gradient-based models (e.g. *TV*), should be investigated.

Another significant aspect in CS development is the recovery algorithm. Though only the  $l_1$  and *TV* minimization were evaluated in this work, there are recent algorithms that not only speed up, but also improve reconstruction [14].

Different quantization models have also been proposed as a way of improving CS performance [5]. Moreover, alternative sensing matrices are also worth investigating.

Finally, it is critical to discuss the universality of CS. Most of the referenced publications point out that one of the greatest advantages of CS is that it does not need to be adaptive. By this, we mean that encoding can be done without the knowledge of the sparsity distribution. In fact, as long as a domain in which the signal is sparse exists, the same random measurements can be taken to reconstruct it. However, in the experiments performed, the best trade-off between the number of measurements and the quantization step varies according to the signal's sparsity distribution. Devising strategies that mitigate this effect is a topic for further investigation.

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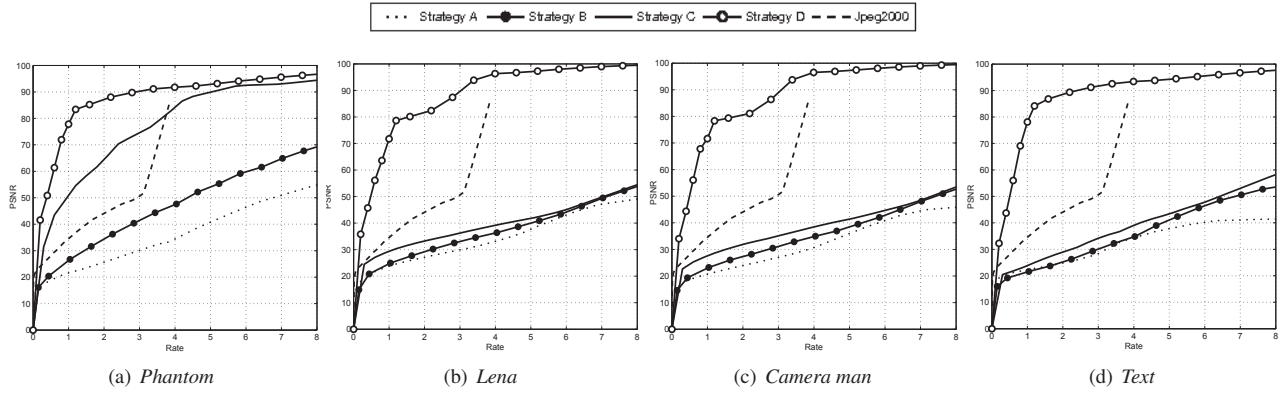


Fig. 2. Rate-Distortion curves for all considered strategies and JPEG2000.

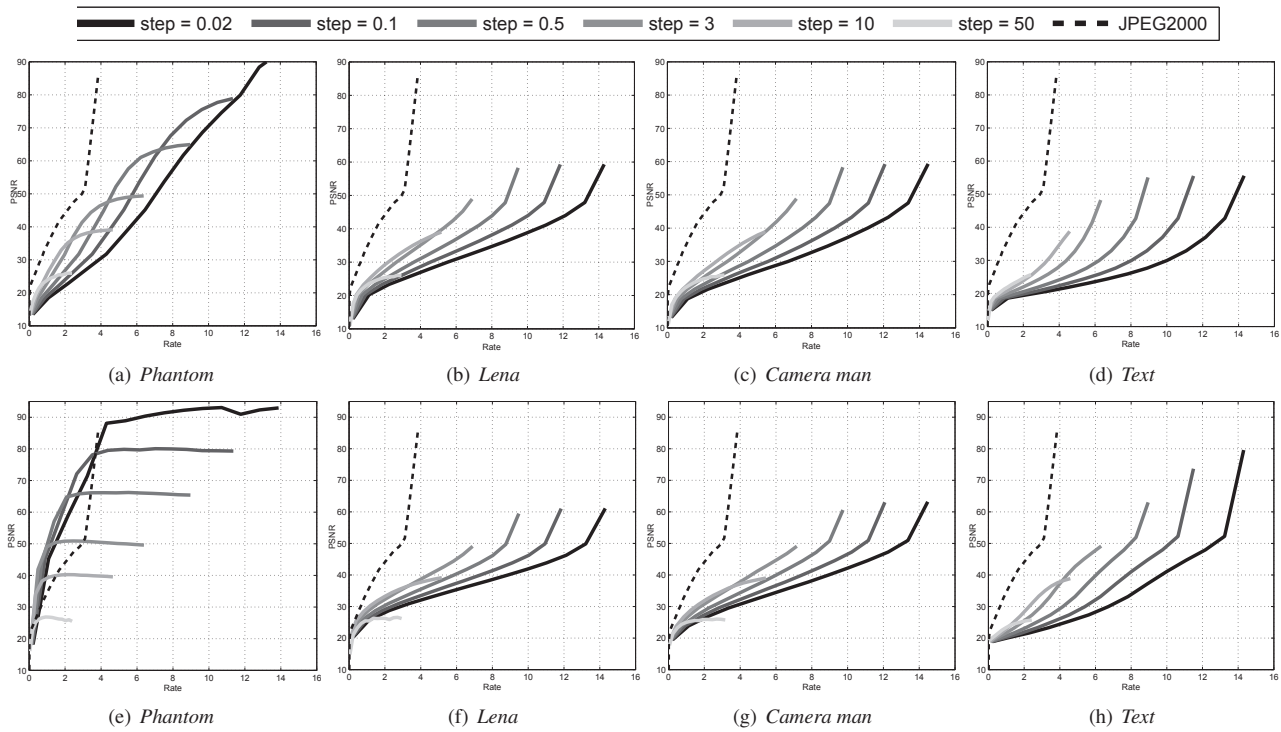


Fig. 3. Rate  $\times$  PSNR for varying quantization steps: (a-d) strategy B and (e-h) strategy C (see Section 2 for strategies definition).

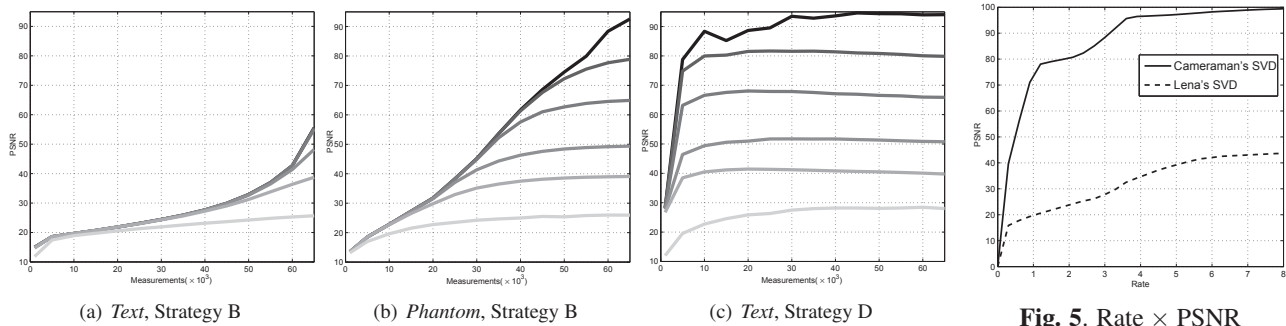


Fig. 4. Number of Measurements  $\times$  PSNR (see Section 2 for strategies definition).

Fig. 5. Rate  $\times$  PSNR Camera man, Strategy D.