On the Coding Gain of Wavelet Transforms

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ABSTRACT

The coding gain of biorthogonal wavelet transforms is investigated. The influence of the regularity and number of vanishing moments of these wavelets on the coding gain is analysed. More than 80 wavelet transforms are assessed. The results show that as long as the coding gain of a wavelet transform is reasonably high, the influence of both regularity and number of vanishing moments on the coding gain is low. On the other hand, for wavelet transforms with coding gains less than 1, the coding gain decreases when the regularity decreases. It is also shown that, in some cases, the coding gain of biorthogonal wavelet transforms is larger than the theoretical maximum for orthogonal transforms, obtained with ideal filters. In addition, unlike orthogonal transforms, whose coding gain in octave decompositions is always smaller than the one obtained with uniform band decompositions with the same number of stages and filter banks, the coding gain of biorthogonal transforms in octave band decompositions can be either larger or smaller than in the uniform case.

BIORTHOGONAL WAVELET TRANSFORMS

A Discrete Biorthogonal Wavelet Transform of a function $x(t)$, represented by the coefficients $\check{x}_{m,n}$, is its decomposition on expansions and translations of a mother function $\overline{\psi}(t)$, such that [1]:

$$x(t) = \sum_m \sum_n \check{x}_{m,n} 2^{-\frac{m}{2}} \overline{\psi}(2^{-m}t - n) \quad (1)$$

$$\check{x}_{m,n} = \int_{-\infty}^{\infty} 2^{-\frac{m}{2}} \psi(2^{-m}t - n) x(t) dt \quad (2)$$

The set of functions $\psi(2^{-m}t - n)$ is orthogonal to the set of functions $\overline{\psi}(2^{-r}t - s)$, i. e. , they comprise a biorthogonal set. If these sets are equal ($\psi(t) = \overline{\psi}(t)$), we have an Orthogonal Wavelet Transform. The functions $\psi(t)$ and $\overline{\psi}(t)$ are called the analysis and synthesis wavelets, respectively.

It can be shown that a discrete wavelet transform can be computed via an octave band subband decomposition, where the filter coefficients are derived from the wavelets $\psi(t)$ and $\overline{\psi}(t)$ [1]. The $\mathcal{Z}$ transforms of the analysis low and high pass filters, $H_0(z)$ and $H_1(z)$, are related to those of the synthesis low and high pass filters, $G_0(z)$ and $G_1(z)$, by [1]:

$$H_0(z)H_1(-z) - H_0(-z)H_1(z) = 2z^{-2m+1} \quad (3)$$

$$G_0(z) = z^{2m-1}H_1(-z) \quad (4)$$

$$G_1(z) = -z^{2m-1}H_0(-z) \quad (5)$$

There are some extra conditions which must be satisfied, too. For example, the wavelet has to be regular. Regularity is a measure of how smooth a wavelet is. Being more specific, if a wavelet is m times differentiable and its m^{th} derivative is Hölder continuous of order α, then its regularity will be $m + \alpha$ [2].

It is shown in [2] that, for a wavelet to have a certain regularity, filters $H_0(z)$ and $G_0(z)$ must have a sufficient number of zeros at $z = -1$. Also, the fact that the analysis low pass filter, $H_0(z)$, has N zeros at $z = -1$ implies that the synthesis wavelet has N vanishing moments [1]. Conversely, if $G_0(z)$ has N zeros at $z = -1$ then the analysis wavelet has N vanishing moments. In [1] it is stated that if $\psi(t)$ has N vanishing moments, any function $x(t)$ which is N times differentiable can be represented through the wavelet transform with a great compression potential (which can be checked easily by Taylor series expansion). Thus it might be desirable for the analysis wavelet to have a large number of vanishing moments.

Wavelet transforms used in the assessment

In image processing applications it is convenient to have linear phase wavelet transforms. Since orthogonal wavelets cannot have linear phase [3], we have designed a number of linear phase biorthogonal wavelet transforms for use in the assessment. As a design criterion, for a given filter order, we have set $H_0(z)H_1(-z)$ to have the maximum possible number of zeros at $z = -1$ in order to maximise the regularity of the wavelets. The products $H_0(z)H_1(-z)$ are factorised in the following form:

$$H_0(z)H_1(-z) = (1 + z^{-1})^{2m} Q_m(z) \quad (6)$$

factor	z^0	z^{-1}	z^{-2}
$u(z)$	1		
$q_{2a}(z)$	1	-4.00000000	
$q_{3a}(z)$	1	-6.00000000	12.666667
$q_{4r}(z)$	1	-3.36953638	
$q_{4a}(z)$	1	-4.63046362	9.59748438
$q_{5a}(z)$	1	-3.79973062	7.82659231
$q_{5b}(z)$	1	-6.20026938	12.0426257
$q_{6r}(z)$	1	-3.18907652	
$q_{6a}(z)$	1	-3.25460184	6.70249763
$q_{6b}(z)$	1	-5.55632164	10.6707341
$q_{7a}(z)$	1	-2.87085009	5.93299839
$q_{7b}(z)$	1	-4.99714255	9.52600936
$q_{7c}(z)$	1	-6.13200736	11.5847200

Table 1: Coefficients of the linear phase factorisation of the several $Q_i(z)$

where linear phase factorisation of the $Q_i(z), i = 2, \ldots, 7$ are as follows (see table 1):

$$Q_2(z) = Q_2(\infty)q_{2a}(z) \quad (7)$$
$$Q_3(z) = Q_3(\infty)q_{3a}(z) \quad (8)$$
$$Q_4(z) = Q_4(\infty)q_{4r}(z)q_{4a}(z) \quad (9)$$
$$Q_5(z) = Q_5(\infty)q_{5a}(z)q_{5b}(z) \quad (10)$$
$$Q_6(z) = Q_6(\infty)q_{6r}(z)q_{6a}(z)q_{6b}(z) \quad (11)$$
$$Q_7(z) = Q_7(\infty)q_{7a}(z)q_{7b}(z)q_{7c}(z) \quad (12)$$

To make reference to the wavelet transforms resulting from a certain factorisation easier, the following notation is assigned to each wavelet transform:

$$n_1.c_1c_2c_3n_2\_c_4c_5c_6n_3 \quad (13)$$

This code means that the corresponding biorthogonal wavelet transform is generated by the following filters:

$$H_0(z) = u(z)^{n2}q_{n1c1}(z)q_{n1c2}(z)q_{n1c3}(z) \quad (14)$$
$$H_1(-z) = u(z)^{n3}q_{n1c4}(z)q_{n1c5}(z)q_{n1c6}(z) \quad (15)$$

where $u(z) = 1 + z^{-1}$.

Using the above notation, the biorthogonal wavelets analysed in this work are: 2.a1_3, 2.a2_2, 3.a2_4, 3.a3_3, 3.a4_2, 4.a2_r6, 4.a3_r5, 4.a4_r4, 4.r1_a7, 4.r2_a6, 4.r3_a5, 4.ra4_4, 4.ra5_3, 4.ra6_2, 5.a3_b7, 5.ab7_3, 5.ab8_2, 5.b3_a7, 5.b4_a6, 5.b5_a5, 5.b6_a4, 6.a2_rb10, 6.ab6_r6, 6.ab7_r5, 6.b3_ra9, 6.b4_ra8, 6.b5_ra7, 6.b6_ra6, 6.r1_ab11, 6.r2_ab10, 6.r3_ab9, 6.r4_ab8, 6.ra4_b8, 6.ra5_b7, 6.rab10_2, 6.rab9_3, 6.rb6_a6, 6.rb7_a5, 6.rb8_a4, 6.rb9_a3, 7.ab6_b8, 7.abc11_3, 7.abc12_2, 7.ac6_b8, 7.ac7_b7, 7.ac8_b6, 7.b3_ac11, 7.b4_ac10, 7.b5_ac9, 7.bc11_a3, 7.c3_ab11, 7.c4_ab10, 7.c5_ab9, 7.c6_ab8, and 7.c7_ab7.

The orthogonal wavelets defined by Daubechies [3], have also been used in the comparison. The QMF filters defined by Johnston [4] are also analysed despite the fact that they do not comprise wavelet transforms in the exact sense (they are not perfect reconstruction filter banks). This is done because they approximate ideal filters reasonably well, and thus they are good approximations for orthogonal wavelet transforms, with the advantage of having linear phase and being standard filter banks.

THEORETICAL PERFORMANCE OF WAVELET TRANSFORMS

In this section the influence of the regularity and number of vanishing moments of wavelet transforms on their performance is assessed. In addition, wavelet transforms (octave band decompositions) are compared against uniform frequency subdivision schemes using the same filter banks. The measure used for performance analysis of the wavelets is the generalised coding gain proposed by Katto and Yasuda [5], which is a generalisation of the coding gain for any perfect reconstruction subband analysis/synthesis system.

Estimates of regularity based on [2] were computed for each of the biorthogonal wavelet transforms designed in the previous section as well as for the Daubechies' wavelets referred to in the previous section. Since the QMF filters are not perfect reconstruction filter banks, they are not included in the regularity analysis. For each biorthogonal and orthogonal wavelet transforms and for the QMF filters as well, the generalised coding gain was computed for octave band decompositions (i. e. , wavelet transforms) with 1 to 5 stages (2 to 6 bands decomposition). For the sake of comparison, the coding gain was also computed for the same filter banks generating the wavelet transforms with uniform band decompositions, for 1 to 5 stages (2, 4, ..., 2^5 bands decomposition).

Figure 1.a shows the behaviour of the coding gain versus the regularity of the analysis wavelet of a 1 stage wavelet transform (2 band decomposition). The graph shows that there is very little correlation between the gain and the regularity of the analysis wavelet. When the number of stages is greater than 1, the correlation tends to increase, as shown in figure 1.b for the 5 stage case. The degree of increase at lower values of regularity of the analysis wavelet is significant. It was observed that even the 5 stage uniform decomposition has a similar behaviour to the octave band decompositions of figure 1.b. Hence, it can be concluded that the regularity of the analysis wavelet can have a strong influence in the coding gain of a multistage decomposition if the wavelets are not regular (regularity less than 1). However, as long as the analysis wavelet is regular enough (regularity greater than 1), its effect on the coding gain seems to be negligible. This result suggests that, as long as the coding gain of the wavelet transforms is concerned, there is no need to aim for analysis wavelets with regularities greater than 1.

Figure 1.c shows the effect of the regularity of the synthesis wavelets on the coding gains for 5 stage wavelet transforms. The correlation is still small, but there is a slight tendency of decreasing the coding gain at high values of regularity. This can be explained by inspecting the filter design process. Since, for a given order, the number of zeros at $z = -1$ is fixed, if we increase the regularity of the synthesis wavelet by setting more zeros at $z = -1$ in the filter $G_0(z)$, these zeros have to be taken away from $H_0(z)$, hence reducing the regularity of the analysis wavelet. Therefore, if the regularity of the synthesis wavelet is too high, the analysis wavelet becomes irregular, and the coding gain is reduced. Thus, it can be concluded that the regularity of the synthesis wavelet also does not play any key role on the coding gain.

In [6], it is stated that in orthogonal systems with a large number of stages the coding gain of an octave band decomposition, as in wavelet transforms, is smaller than that of a uniform decomposition. The difference in coding gains between 5 stage uniform and octave band decompositions for all the filter banks considered, including the QMF ones, is shown in figure 1.d. As can be seen, the statement of [6] is also true for QMF filter banks, which are quasi orthogonal, but not valid for biorthogonal wavelet transforms. It depends on the type of wavelet transforms, as for some, the coding gain of octave band decompositions may be higher or lower than that of uniform band decompositions. This occurs specially at high values of coding gain, as shown in figure 1.e. At low coding gains, uniform band decompositions have in general smaller coding gains than octave band decompositions. In addition, for 1 and 2 stage wavelet transforms, some biorthogonal filters have coding gains higher than the theoretical upper bound for orthogonal systems, obtained with ideal filters. For example, the coding gain of the biorthogonal wavelet transform 6.rb7_a5 for 1 stage decomposition is 6.78dB, almost 1dB higher than the one obtained with ideal filters, 5.96dB [7]. The coding gain of the biorthogonal wavelet transform 7.c6_ab8 for 2 stage octave decomposition is 8.90dB, still higher than the one obtained with the ideal filters and uniform decomposition, 8.59dB, which is the theoretical upper bound for orthogonal systems with the same number of stages. Figure 1.f shows the influence of the number of vanishing moments of the analysis wavelets on the coding gain of a 5 stage wavelet transform. It seems that the number of vanishing moments of the analysis wavelet has no correlation with the coding gain. A similar behaviour was observed for the number of vanishing moments of the synthesis wavelet. From this, it can be concluded that, despite the fact that a large number of vanishing moments of the analysis wavelet might lead to an efficient representation of sufficiently smooth functions in a "Taylor series" sense [1], its influence on the coding gain is small.

CONCLUSIONS

In this paper an extensive number of wavelet transforms was analysed. The influence of wavelet features such as regularity and the number of vanishing moments on the coding gain was investigated. It has been verified that as long as the regularity of the analysis wavelet is larger than 1, the increase in regularity does not enhance the coding gain of a wavelet transform. On the other hand, if the regularity is less than 1, then the lower is the regularity the smaller becomes the coding gain.

In biorthogonal wavelet transforms the coding gain of octave band decompositions can be either smaller or larger than that of uniform band decompositions with equal number of stages and the same filter banks. This is different from the orthogonal case, where the coding gain of uniform band decompositions is always larger than the one of octave band decompositions. In addition, it has also been shown that, in some cases, biorthogonal wavelet transforms can have a higher coding gain than the theoretical maximum for orthogonal systems, obtained with ideal filter banks.

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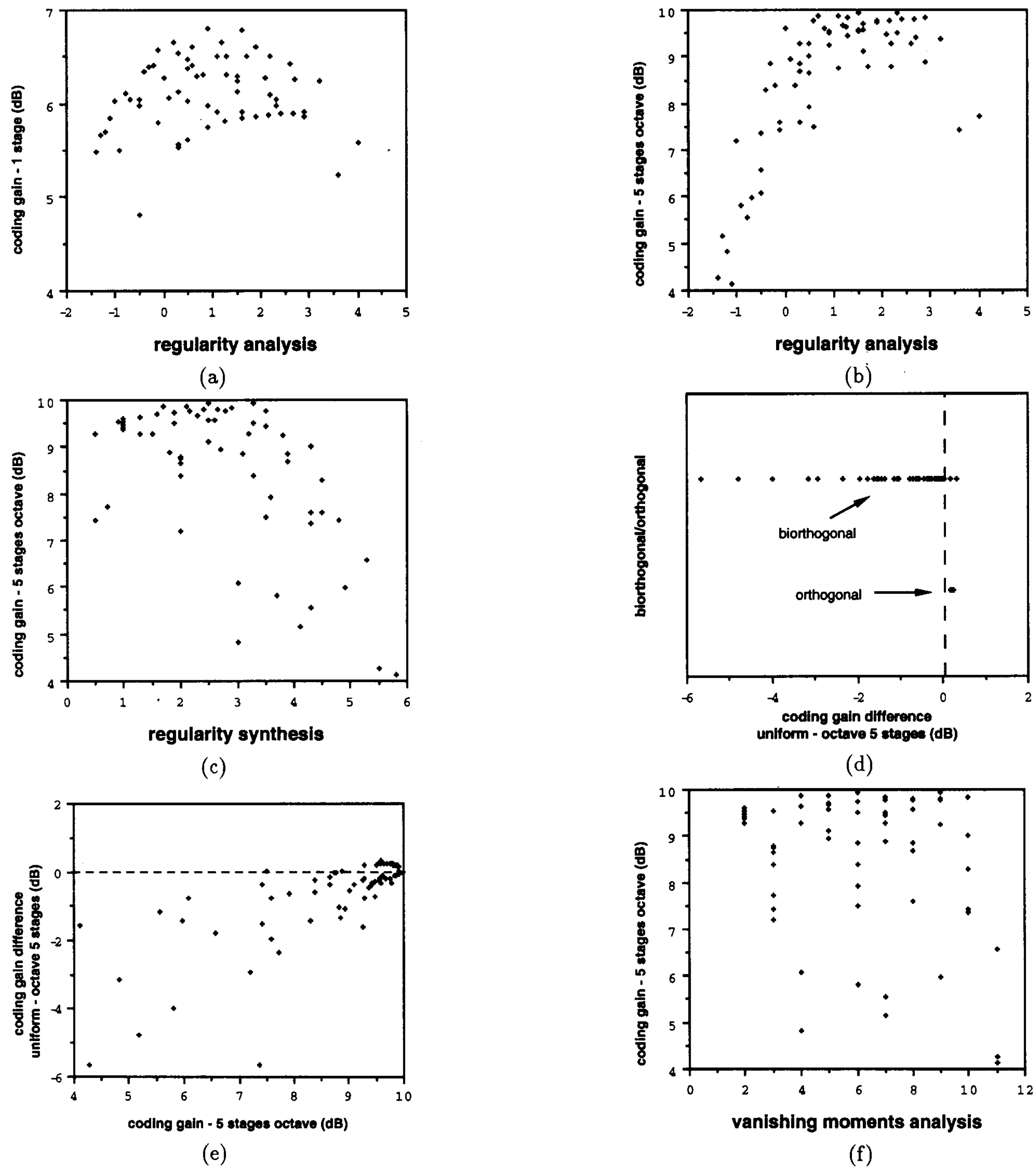

Figure 1: Regularity of the analysis wavelet versus: (a) coding gain 1 stage; (b) coding gain 5 stage octave; (c) Regularity of the synthesis wavelet versus coding gain 5 stage. Differences between the coding gains of the uniform and octave band decompositions: (d) for orthogonal and biorthogonal wavelets; (e) versus the coding gain of a 5 stage wavelet transform. (f) Coding gain of a 5 stage octave decomposition versus the number of vanishing moments of the analysis wavelet