Low-Sensitivity Digital-Filter Structures which are Amenable to Error-Spectrum Shaping

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Abstract—A systematic and exhaustive procedure is used to generate a class of new low-sensitivity second-order digital-filter structures which are amenable to error-spectrum shaping (ESS). Collectively, these structures can realize any stable second-order transfer function. For every transfer function, a choice of at least two structures is available but, through a sensitivity analysis, the optimum one can always be identified. The paper concludes with sensitivity and roundoff-noise comparisons which show that the new structures are usually superior relative to corresponding direct canonic structures and also relative to the state-space section-optimal structure.

I. INTRODUCTION

ROUNDOFF noise can be reduced in recursive digital filters by increasing the wordlength [1], by choosing the structure appropriately [2], [3], or by applying error-spectrum shaping (ESS) [4]–[8].

ESS is a quantization technique which involves the generation of an error signal and the application of local feedback for the purpose of forcing zeros in the power spectral density of the output noise. The technique can be implemented by incorporating a quantizer and a corresponding substructure between the output and input of each and every adder whose inputs include at least one nontrivial product (a signal multiplied by a noninteger constant). Therefore, the application of ESS entails an increase in the complexity of hardware. Furthermore, the complexity of hardware increases as the number of adders of the type described is increased and, consequently, the application of this technique has been restricted to direct cascade realizations in the past [4]–[8].

The application of ESS to direct realizations tends to bring about a dramatic reduction in the output roundoff noise [4]–[8]. Unfortunately, however, the sensitivity to coefficient quantization is not affected by ESS and, as is well known, it can be large, in particular if the poles of the transfer function are close to the unit circle of the z plane [9], [10].

In this paper, a systematic procedure is described which can be used for the generation of low-sensitivity digital-filter structures which are amenable to ESS. The procedure is used to generate two sets of structures which include several new structures as well as some known structures

Fig. 1. General structure which is amenable to ESS.

like those of Agarwal and Burrus [9], and Nishimura, Hirano, and Pal [10]. The emphasis is placed on generating second-order structures which can be used in cascade or in parallel for the realization of high-order transfer functions.

II. SYNTHESIS PROCEDURE

The application of ESS in a second-order structure is economically attractive only if product quantization can be achieved by using no more than one quantizer. Consequently, a second-order structure is amenable to ESS only if all nontrivial signal-coefficient products in the structure are inputs to only one adder.

A general second-order structure can be constructed as depicted in Fig. 1 where $y_0(n)$, $y_1(n)$, and $y_2(n)$ are internal node signals which will be used later for the placement of zeros. If coefficients m_i are assumed to be noninteger constants and substructure N is assumed to be free of noninteger multipliers, a general second-order structure is obtained which is amenable to ESS. By applying a systematic and exhaustive search to the general structure of Fig. 1, all possible low-sensitivity structures can be generated and appropriate design formulas can be deduced.

In this section, the above approach is used for the realization of all-pole transfer functions. The realization of biquadratic transfer functions is accomplished through zero-placement techniques and is considered in Section III.

The simplest structure that can realize any second-order allpole transfer function must have three nodes [11], where each node may represent a distribution or a summation point. The most general structure of this class will have forward and reverse paths between any two of the three nodes and, further, either the forward or reverse path in each case must correspond to a unit delay in order to avoid delay-free loops. On this basis, the simplest form of the general structure of Fig. 1 is obtained, as shown in Fig. 2,

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Fig. 2. Second-order structure of low complexity which is amenable to ESS.

where branches A , B , C , D , and E represent unit delays or integer multipliers whose multiplier constants are restricted to $0, \pm 1, \pm 2$. Evidently, a forward path has not been included between the input adder and node $y_2(n)$ since this would necessitate a third unit delay.

The structure of Fig. 2 realizes the characteristic polynomial

$$\begin{aligned} D(z) &= (1 - BD - AC - m_1A + ABE + m_2AB \\ &\quad + ABCD + m_1ABD)z^2 \\ &= z^2 + \alpha_1z + \alpha_2 \\ &= (z - re^{j\omega_0})(z - re^{-j\omega_0}) \end{aligned} \quad (1)$$

where

$$r = \sqrt{\alpha_2}, \quad \omega_0 = \cos^{-1}(-\alpha_1/2\sqrt{\alpha_2}).$$

In order to avoid delay-free loops [1] and maintain the number of delays to the minimum of two, the constraints

$$A = z^{-1}, \quad \text{and} \quad B \text{ or } D = z^{-1} \quad (2)$$

must be satisfied. Therefore, two cases are possible, as follows:

Case I: $A = B = z^{-1}$

Case II: $A = D = z^{-1}$

Case I

For Case I, the characteristic polynomial of (1) becomes

$$D(z) = z^2 - z(C + D + m_1) + CD + m_1D + m_2 + E. \quad (3)$$

In order to obtain low sensitivity, multipliers C , D , and E must be chosen as

$$C + D = \text{Int}[-\alpha_1] \quad (4)$$

$$E = \text{Int}[\alpha_2 + \alpha_1D + D^2] \quad (5)$$

where $\text{Int}[x]$ is the closest integer to x . Equations (4) and (5) force the values of m_1 and m_2 to be low and, as demonstrated in [9], this assures low sensitivity of the transfer function with respect to changes in m_1 and m_2 for the case where floating-point arithmetic is used or for the case where fixed-point arithmetic is used but the coefficients are stored in normalized floating-point form.

The choice of structure tends to depend heavily on the pole positions. Therefore, several possibilities must be ex-

TABLE I
STRUCTURES FOR CASE I

STRUCTURE	COEFFICIENTS			RANGE OF α_1
	D	C	E	
I-1	1	1	0	$-2.0 < \alpha_1 < -1.5$
I-2	0	2	1	
I-3	2	0	1	$-2.0 < \alpha_1 < -1.75$
I-4	2	0	2	$-1.75 < \alpha_1 < -1.5$
I-5	0	1	1	$-1.5 < \alpha_1 < -0.5$
I-6	1	0	1	
I-7	0	0	1	$-0.5 < \alpha_1 < 0.5$
I-8	1	-1	2	
I-9	-1	1	2	$0.5 < \alpha_1 < 1.5$
I-10	0	-1	1	
I-11	-1	0	1	$1.5 < \alpha_1 < 1.75$
I-12	-2	0	2	
I-13	0	-2	1	$1.5 < \alpha_1 < 2.0$
I-14	-1	-1	0	
I-15	-2	0	1	$1.75 < \alpha_1 < 2.0$

amined. If the poles are close to $z = 1$, then $\alpha_1 \cong -2$ and $\alpha_2 \cong 1$, and so

$$C + D = 2.$$

We can thus assign

$$C = 1, \quad D = 1 \quad \text{and} \quad E = 0$$

This choice of coefficients yields a structure due to Agarwal and Burrus [9], which is suitable for values of α_1 in the range $-2.0 < \alpha_1 < -1.5$. It will be referred to as structure I-1. Alternatively, if

$$C = 2, \quad D = 0 \quad \text{and} \quad E = 1$$

another structure due to the same authors [9] is obtained. This structure is suitable for the same range of α_1 as structure I-1 and will be referred to as structure I-2. A third possibility is obtained by letting

$$C = 0, \quad D = 2 \quad \text{and} \quad E = 1$$

This structure will be referred to as structure I-3, and is to our knowledge new. It is suitable for values of α_1 in the range $-2.0 < \alpha_1 < -1.75$.

By considering poles which are close to the unit circle $|z| = 1$, several new structures can be obtained for different ranges of values of α_1 , as shown in Table I.

Case II

For Case II, the characteristic polynomial of (1) becomes

$$D(z) = z^2 - z(B + C + m_1 - m_2B - BE) + BC + m_1B. \quad (6)$$

In order to obtain low sensitivity, multiplier coefficients B ,

TABLE V
NUMERATOR AND DENOMINATOR POLYNOMIALS OF SENSITIVITIES

Case	$N_{m_1}(z)$	$N_{m_2}(z)$	$D_{m_1}(z)$
I	$m_1(z-D)$	$-m_2$	$D(z)$
II	$m_1(z-B)$	$-m_2Bz$	$D(z)$

constant, then for ω close to the pole frequency ω_0 (see (1))

$$|S_{m_i}^{M(\omega)}| \gg |m_i| \left| \frac{\partial \theta(\omega)}{\partial m_i} \right|$$

and hence

$$|S_{m_i}^{H(e^{j\omega T})}| \approx |S_{m_i}^{M(\omega)}|. \quad (14)$$

In addition, if m_j is a numerator multiplier constant then

$$|S_{m_i}^{H(e^{j\omega T})}| \gg |S_{m_j}^{H(e^{j\omega T})}| \quad (15)$$

and, therefore, for $\omega \approx \omega_0$ (13)–(15) yield

$$S \approx \sum_{i=1}^{k_d} |S_{m_i}^{H(e^{j\omega T})}| \approx \sum_{i=1}^{k_d} |S_{m_i}^{M(\omega)}| \quad (16)$$

where k_d is the number of denominator noninteger multiplier constants. This quantity can serve as a sensitivity measure which can be used for the comparison of different high-selectivity structures.

As was demonstrated in Section II, several structures are possible in which ESS can efficiently be applied. Furthermore, for each permissible range of α_1 at least two distinct structures are possible, as can be seen in Tables I and II. It would thus be a useful exercise to identify an optimal subset of these structures.

The value of S at a specific frequency and its average value over the baseband are strongly related to the maximum value of S , denoted by $\hat{S}$, which occurs at $\omega = \omega_0$. It is thus possible to identify the optimal structures in Tables I and II by using $\hat{S}$ as a sensitivity measure.

For $-1.5 \leq \alpha_1 < -0.5$, the structures to be compared are I-5, I-6, and II-2. Assuming that α_2 is close to unity, (12) and (16), and Table V yield the maximum sensitivities of these structures as

$$\hat{S}_{I-5} \approx \frac{N_{I-5}}{|D(e^{j\omega_0 T})|}$$

$$\hat{S}_{I-6} \approx \frac{N_{I-6}}{|D(e^{j\omega_0 T})|}$$

$$\hat{S}_{II-2} \approx \frac{N_{II-2}}{|D(e^{j\omega_0 T})|}$$

where

$$N_{I-5} = 1 - \alpha_2 + |1 + \alpha_1|$$

$$N_{I-6} = |\alpha_1 + \alpha_2| + |1 + \alpha_1| \sqrt{(2 - 2 \cos \omega_0)}$$

$$N_{II-2} = |\alpha_1 + \alpha_2| + (1 - \alpha_2) \sqrt{(2 - 2 \cos \omega_0)}.$$

For $-1.5 \leq \alpha_1 < -1$ and $\alpha_2 < 1$

$$N_{I-5} = -(\alpha_1 + \alpha_2)$$

Fig. 5. Sensitivity versus frequency: — Structure I-6, ---- Structure I-5, - · - · - Structure II-2.

$$N_{I-6} = -(\alpha_1 + \alpha_2) - (1 + \alpha_1) \sqrt{(2 - 2 \cos \omega_0)}$$

$$N_{II-2} = -(\alpha_1 + \alpha_2) + (1 - \alpha_2) \sqrt{(2 - 2 \cos \omega_0)}$$

and hence

$$\hat{S}_{I-5} < \min \{ \hat{S}_{I-6}, \hat{S}_{II-2} \}.$$

For $-1 \leq \alpha_1 < -\alpha_2$,

$$N_{I-5} = 2 - \alpha_2 + \alpha_1 \quad (17)$$

$$N_{I-6} = -(\alpha_1 + \alpha_2) + (1 + \alpha_1) \sqrt{(2 - 2 \cos \omega_0)} \quad (18)$$

$$N_{II-2} = -(\alpha_1 + \alpha_2) + (1 - \alpha_2) \sqrt{(2 - 2 \cos \omega_0)}. \quad (19)$$

Evidently,

$$N_{I-6} < N_{II-2}$$

and, in addition

$$N_{I-6} < N_{I-5}$$

as can be shown by adding $(\alpha_1 + \alpha_2)$ to each of (17) and (18), and then dividing each by $(1 + \alpha_1)$. As a consequence

$$\hat{S}_{I-6} < \min \{ \hat{S}_{I-5}, \hat{S}_{II-2} \}.$$

Similarly, for $-\alpha_2 \leq \alpha_1 \leq -0.5$

$$N_{I-5} = 2 - \alpha_2 + \alpha_1$$

$$N_{I-6} = (\alpha_1 + \alpha_2) + (1 - \alpha_1) \sqrt{(2 - 2 \cos \omega_0)}$$

$$N_{II-2} = (\alpha_1 + \alpha_2) + (1 - \alpha_2) \sqrt{(2 - 2 \cos \omega_0)}$$

and as above we can show that

$$\hat{S}_{II-2} < \min \{ \hat{S}_{I-5}, \hat{S}_{I-6} \}. \quad (20)$$

In effect, the choice of structure tends to depend heavily on the relative values of coefficients α_1 and α_2 . For example, if $\alpha_1 = -0.9$ and $\alpha_2 = 0.995$, I-5, I-6, and II-2 are possible structures, and according to (20) the optimal one is II-2. This result is confirmed by the sensitivity plots in Fig. 5.

Fig. 7. Amplitude responses. (a) Low-pass filter, 8 bits. (b) High-pass filter, 5 bits. (c) Bandpass filter, 7 bits. (d) Bandstop filter, 5 bits. — Ideal amplitude response, ---- Section-optimal structure, — · — Direct-canonic structure, — — — New structure (II-1 for lowpass filter, II-6 for high-pass filter, I-5 for bandpass filter, I-7 for bandstop filter).

tures, ESS has been applied using the formulas in [8]. The quantization of products was carried out after the adders for all sections in the cascade except for the numerator multipliers of the output section where the quantization was carried out before the adder. ESS has not been applied in the corresponding designs based on the section-optimal structure since the number of multiplications would then become excessive. It should be mentioned that the proposed structures with ESS incorporated require between five and seven multipliers per section whereas the section-optimal structure without ESS requires nine multipliers per section.

The results of the roundoff-noise analysis are depicted in Fig. 8(a)–(d). As can be seen, for the low-pass, high-pass, and bandpass filters, the proposed structures with ESS incorporated lead to superior results, in particular, if sec-

ond-order ESS is applied. For the bandstop filter, on the other hand, the passband average of the RPSD is lower in the section-optimal structure whereas the passband maximum of the RPSD is lower in the proposed structure. It should be mentioned that a similar reduction in output roundoff noise can be achieved by applying ESS to direct canonic structures. However, as was pointed out earlier, these structures are almost always more sensitive relative to the proposed structures.

VII. CONCLUSIONS

A systematic and exhaustive procedure has been described for the generation of economical low-sensitivity digital-filter structures which are amenable to the application of ESS. The procedure has then been used to generate the structures of Tables I and II.

Fig. 8. Output-noise spectra. (a) Low-pass filter. (b) High-pass filter. (c) Bandpass filter. (d) Bandstop filter: ----- Section-optimal structure, ——— New structure with 1st-order ESS, ——— New structure with 2nd-order ESS (II-1 for low-pass filter, II-6 for high-pass filter, I-5 for bandpass filter, I-7 for bandstop filter).

In all, 21 structures were obtained and except for structures I-1, I-2, and II-1, which were reported in [9], [10], they are thought to be new. For each range of α_1 , at least two distinct structures are possible but, through the sensitivity analysis of Section IV, the optimum structure for the transfer function under consideration can easily be chosen, as shown in Table VI.

A sensitivity comparison has shown that the proposed structures are usually superior relative to corresponding direct canonic structures and also relative to the section-optimal structure.

A roundoff noise comparison has shown the proposed structures with ESS incorporated to be superior relative to the section-optimal structure, except in the case of bandstop filters where a lower passband average of the RPSD can be achieved by using the section-optimal structure.

Further, despite the additional multipliers needed for ESS, these structures require fewer multipliers per second-order section.

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